

The general solution is

$$C(\ell) = \chi \left(\frac{\log \ell}{\log \sigma} \right) \ell^d \quad (3.7)$$

where χ is a periodic function of period one. The general solution shows the possibility of oscillations observed in the scaling of strictly self-similar sets and in many numerical experiments. Identifying $1/\sigma$ with the similarity ratio, we find agreement of the periodicity with the textbook examples.

In strictly self-similar fractals the presence of oscillations is assured. In more general fractals, scaling is expected only in an asymptotic sense. The observed oscillations are first corrections to the $d_0 \log \ell$ term in the series for $C(\ell)$; that is, as ℓ approaches zero, we have

$$\log C(\ell) = d_0 \log \ell + \psi(\log \ell / \log \sigma) + \dots \quad (3.8)$$

where ψ is a periodic function of period one. In inhomogeneous fractals the double average in the correlation integral may tend to wash out the oscillations at large scales if the lacunarity is not uniform over the set. For instance, at $\ell \approx 1$ (standard units) in the Hénon attractor, $C_2(\ell)$ is not dominated by the frequently shown Cantorial sections, but by the more heavily weighted gossamer (linear) portions; the amplitude of ψ is small. A deviation from a smooth power law relationship between the number of grid boxes that contain a portion of the attractor and the box size at small scales

has been observed by Cawley and York (1986, preprint), who point to the need for good error determination. Following their suggestion, we will indicate the range of ℓ considered when an exponent is determined by a linear least square fit of $\log C(\ell)$ on $\log \ell$.

In the limit of small ℓ , $C_2(\ell)$ should be governed by the oscillation of the most frequently visited region of the attractor. One advantage of examining $C_1(x, \ell)$ is that the fine structure of a region is immediately available without the interference of other portions of the attractor. For strictly self-similar, homogeneous fractals such as the Cantor sets above, the phase of the oscillation is locked in place by the stringent requirements of homogeneity. In those instances, the oscillation is easily accessible in both $C_2(\ell)$ and $C_1(x, \ell)$.

In addition to providing a new handle on the physics of a problem, the presence of an oscillation holds practical consequences in calculating scaling exponents. These are most strongly expressed when the amplitude of the oscillation is large and its period is long compared to the available data. At such times the oscillation introduces a strong bias into the measurement; a good example is found in the Zaslavsky attractor (Zaslavsky, 1978; see also Badii and Politi, 1985 and Grassberger and Procaccia, 1983b). It is not possible to determine the exponent of a data set via standard techniques unless the range of available data is greater than the period of the oscillation. When the range is less than a period, the exponent may appear to be "scale dependent"; such an insufficiently resolved fractal set may be confused with a multi-fractal set (see

Section 4.4).

Section 3.3: Oscillatory Component of Scaling in Inhomogeneous Fractals

In the next two sections we determine μ for two inhomogeneous fractal sets. While we shall use the Feigenbaum set and the Hénon attractor as examples, the method should be useful in the analysis of many fractal sets and attractors in general, and in the analysis of some experimental results. In the latter, its usefulness will vary with the quality of the reconstruction. At present, most attractors which are reconstructed from experimental time series are formed using a delay time related to the first zero of the autocorrelation function. When this method is used to determine the delay time, the reconstruction process tends to squash the folded portions of the attractor. As Fraser and Swinney (1986) have pointed out, the autocorrelation reflects the linear dependence of the data, while the mutual information measures the general dependence. (See Fraser and Swinney, 1986 for definitions and an application to experimental data.) Their alternative method of choosing a delay time by minimizing the mutual information yields a reconstruction with a much more self-similar appearance. We conjecture that an improved reconstruction will expose the presence of the oscillation in many physical systems. Judging from Figure 1 of Fraser and Swinney (1986), the oscillation may already be visible in the data from the Belousov-Zhabotinskii reaction.

Our first example is generated by the logistic map

$$x_{i+1} = 4\lambda x_i (1-x_i). \quad (3.9)$$

There are many introductions to this mapping, due to its widespread application. See, for instance, May (1986), Feigenbaum (1980), and Cvitanović (1983). A general treatment of maps on the unit interval is given by Collet and Eckmann (1980). The familiar bifurcation tree for this map is shown in Figure 3.4. This plot is constructed by iterating the map 2^{10} times starting from $x_0 = \frac{1}{2}$ for each of 800 values of λ on the interval $0.625 \leq \lambda \leq 1.0$. Treating the first 2^8 values of x as transient, we plot the remaining points. The map shows "attracting" orbits of varying period with λ . The values at which the orbital period doubles from 2^n to 2^{n+1} are related geometrically by

$$\delta_F = \lim_{n \rightarrow \infty} \frac{\lambda_{n-1} - \lambda_n}{\lambda_n - \lambda_{n+1}} \approx 4.6692016 \quad (3.10)$$

while ratio of successive branch splittings (see Figure 3.4) at λ_n and λ_{n+1} scale as

$$\alpha_F = \lim_{n \rightarrow \infty} \frac{\epsilon_n}{\epsilon_{n+1}} \approx 2.502907875 \quad (3.11)$$

The infinite period orbit at $\lambda = \lambda_\infty \approx 0.837005134$ which contains the point $x = \frac{1}{2}$ is known as the Feigenbaum set. Interest in this set comes, in part, from the many observations of period doubling in nature; Cvitanović (1983, Table 9.1) tabulates the values of α_F and

δ_F obtained from experimental observations of ten systems. Libchaber *et al.* (1986) have measured $f(\alpha)$ for the equivalent set reconstructed from a convection experiment and found close agreement with that of the logistic map.

The Hausdorff dimension of the Feigenbaum set has been calculated by Grassberger (1982) as

$$d_{HB} = 0.5381 \pm 0.0006 \quad .$$

Hentschel and Procaccia (1983) calculate the generalized dimensions

$$d_0 = 0.537 \quad d_1 = 0.518 \quad d_2 = 0.501 \quad ,$$

while Grassberger and Procaccia (1983) report

$$\nu = d_2 = 0.500 \pm 0.005 \quad .$$

Our calculation of $C_2(\ell)$ for a 2^{13} point approximation of the Feigenbaum set yields a linear least squares fit of $\log C_2(\ell)$ to $\log(\ell)$ with slope

$$\nu = 0.496 \quad \text{for } 10^{-3} < \ell < 10^{-3} \alpha^6 .$$

Structure in the departure from linear scaling is shown in Figure 3.5a where we plot $(\log C_2(\ell) - \nu \log \ell)$ as a function of $\log \ell$. The first 8 iterations are shown in Figure 3.5b. Since the pattern of future iterates is known, Halsey *et al.* (1986) compute

$$d_{+\infty} = \frac{\log 2}{\log \alpha_F^2} \approx 0.37775$$

and

$$d_{-\infty} = \frac{\log 2}{\log \alpha_F} \approx 0.75551 \quad .$$

(3.12)

In obtaining the results below, we analyzed a 2^{19} point approximation of the set. $C_1(\frac{1}{2}, \ell)$ is shown in Figure 3.6a. Here we see regular oscillation with a period approximately equal to α_F . Removing the mean slope over scales $(0.1/\alpha_F^6)$ to 0.1 yields the oscillation shown in Figure 3.6b, and a value

$$\mu(x_0) = 0.754. \quad ((.1/\alpha_F^6) < \ell < .1) \quad .$$

Since $x_0 = \frac{1}{2}$ is located in the least dense region of the set, we expect $\mu(x_0) = d_{-\infty}$. Next, we consider the densest region of the set which is located near the first iterate of x_0 under the map, $x=x_1$: $C_1(x_1, \ell)$ is shown in Figure 3.7; the oscillation here has period α^2 . This is a result of the flipping of the set between adjacent factors of α_F ; for an identical orientation of the set, two generations are required (see Feigenbaum, 1980).

The oscillatory component of $C_1(x_1, \ell)$ is shown in Figure 3.7b, where the linear component removed had slope

$$\mu(x_1) = 0.370 \quad ((.1/\alpha_F^6) < \ell < .1) \quad .$$

Similarly

$$\mu(x_2) = 0.380 \quad .$$

Here we note $\mu(x_1) \approx d_{+\infty}$, a reasonable correspondence in that x_1 is in the densest region of the set. Now consider x_i for $i > 2$, at scales

$$\ell_y \lesssim (1/\alpha)^a \quad (3.13)$$

where

$$a = [\frac{1}{2} \log(i) + 1].$$

Since x_i is located at an end point of the local image of the set, it will scale as either x_1 or x_2 , both of which have $\mu \approx d_{+\infty}$. By understanding the manner in which these regions are coupled, and the crossover between the two, one should be able to calculate d_2 directly from $d_{+\infty}$.

Section 3.4: Observing the Oscillatory Component in a Strange Attractor

In this section we examine the local scaling structure of the Hénon attractor. Following a brief review of the mapping, the local correlation integral is evaluated about an unstable periodic point of the mapping. The periodicity of the ψ function is then used to determine a μ at this point.

The Hénon mapping represents the most general one-to-one quadratic mapping with constant Jacobian. The map is given by

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + y_n \\ y_{n+1} &= b x_n \end{aligned} \quad (3.14)$$

and its Jacobian is:

$$J = \begin{pmatrix} \delta(x_{i+1}, y_{i+1}) \\ \delta(x_i, y_i) \end{pmatrix} = -b.$$

This system has been most thoroughly studied for the parameter values $a=1.4$ and $b=0.3$. For this parameter choice $|J| < 1$; thus, areas contract under the action of the map. There are no attracting fixed points, yet many initial conditions remain near the origin indefinitely; it is generally believed that the long-time evolution of these initial conditions is governed by a strange attractor. 2^{20} iterations of such an initial condition are shown in Figure 3.8a. The region centered on the unstable fixed point discussed below is shown in increasing detail in 3.8b-e. The large scale structure of the attractor is shown in Figure 1.10.

Once a mapping is chosen and the parameter values are set, it remains to pick a value of $\mathbf{x}$ about which to evaluate $C_1(\mathbf{x}, \ell)$. A natural choice is a fixed point of the mapping. Fixed points satisfy

$$x^* = 1 - ax^{*2} + y^*,$$

$$y^* = bx^*$$

or

$$x^* = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a},$$

$$y^* = bx^*.$$

Let x_+ denote the fixed point whose location is given by the positive root and x_- the point corresponding to the negative. For our choice of parameters, both fixed points are real. The stability of a fixed point is determined by linearizing the map about that point and determining whether small displacements grow. For the Hénon map

$$\begin{pmatrix} \delta x^1 \\ \delta y^1 \end{pmatrix} = \begin{pmatrix} -2ax^* & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} \delta x^0 \\ \delta y^0 \end{pmatrix}$$

where δx^0 and δy^0 represent the initial deviation from (x^*, y^*) and a superscript 1 indicates the value of the variable after one iteration. Solving for the eigenvalues, we find

$$\begin{aligned} \lambda_1 &= -ax^* + \sqrt{a^2 x^{*2} + b} & \hat{e}_1 &= \begin{pmatrix} 1 \\ \frac{b}{\lambda_1} \end{pmatrix} \\ \lambda_2 &= -ax^* - \sqrt{a^2 x^{*2} + b} & \hat{e}_2 &= \begin{pmatrix} 1 \\ \frac{b}{\lambda_2} \end{pmatrix}. \end{aligned}$$

Consider x_+ where

$$x^* = 0.63135448\dots$$

$$y^* = 0.18940634\dots$$

We obtain

$$\lambda_1 = 0.15594632\dots$$

$$\lambda_2 = -1.92373886\dots$$

and

$$\hat{e}_1 = \begin{pmatrix} 1.0 \\ -\lambda_2 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 1.0 \\ -\lambda_1 \end{pmatrix}$$

where we have used the fact that

$$\lambda_1 \lambda_2 = -b.$$

Since $\lambda_2 < -1$, $\mathbf{x}_+$ is unstable and a point will alternately move from one side of $\mathbf{x}_+$ to the other, relative to $\hat{e}_2$. Almost all initial conditions near $\mathbf{x}_+$ will move away along $\hat{e}_2$ increasing their distance from $\mathbf{x}_+$ by a factor of $|\lambda_2|$ per iteration. At $\mathbf{x}_+$, the unstable manifold lies along $\hat{e}_2$. Since $|\lambda_1|$ is less than one, there exists a stable manifold which lies along $\hat{e}_1$. Points which fall exactly on the stable manifold approach $\mathbf{x}_+$ asymptotically. A similar analysis shows $\mathbf{x}_-$ is also unstable.

In Figure 3.9 we present the results of a calculation of $C_1(\mathbf{x}, \ell)$ using a 2^{24} iteration run. The mean scaling behavior and oscillation are evident. The flattening of the curve at small ℓ indicates that there is insufficient data to describe the attractor at separations ℓ less than 10^{-4} . A least squares fit to the data over the range $(10^{-4} < \ell < 0.06)$ provides an initial estimate $\mu = 1.319$. Removing this trend yields the first approximation of $\psi(\ell)$, which is shown in Figure 3.10. Inspecting the residual we suspect that μ has been

underestimated due to a bias from the small ℓ points. The error in μ is minimized by fitting an integer number of oscillations. In general, a function of the delayed difference, $f(\psi(\ell) - \psi(\ell - \pi))$, may be used to obtain an estimate of the period; in this case, the period corresponds to the eigenvalue of the stable manifold of the fixed point. Once the linear features of the attractor are established by the dynamics (Figure 2.8a), one iteration of the map acting on this pattern will produce a similar pattern reduced by a factor of λ_1 and stretched longitudinally by a factor of λ_2 .

Cawley and Yorke (1986 preprint) have pointed out that meaningful error estimates for dimension/exponent estimates are difficult to obtain. They show that the residual of the least squares fit is a poor indicator of the goodness of the estimate. We now see why this is true: the residual is a combined measure of the mean scaling and the magnitude of $\psi(\ell)$. Since the residual does not discriminate between the two, a set with a $\psi(\ell)$ function of large amplitude-- such as the Zaslavsky attractor or the 100010101 Cantor set² -- will produce a "poor fit" when exact values of the scaling exponents are used. Worse still, if a small portion of the oscillation is

² In this Cantor set there are large bands of $\log \ell$ in which no point pairs are separated by ℓ ; these correspond to the "000" segments. In such a band the set scales as if it had dimension zero. Alternating regularly with these regions (in $\log \ell$) are "overpopulated" bands, so that the scaling over factors of the similarity ratio has exponent equal to the Hausdorff dimension. Similarly, the Zaslavski attractor resembles a collection of linear bars of finite width; it scales with exponents near 1 at the outer scales, near 2 when at scales comparable to the width of the bars, and then near 1 again when the structure of individual bars is observed. In this case, the structure at smaller scales has not yet been confirmed.

analyzed, this residual may be (arbitrarily) small, while the estimate of the scaling exponent is incorrect by a great deal. This is a fundamental difficulty in the determination of scaling exponents; for a reliable estimate to be made an indication of the of the structure of smaller scales is needed. Although not conclusive, an oscillatory component provides such evidence. The oscillations are robust, and their appearance is an indicator that the scaling region has been reached.

To obtain a qualified estimate of μ , we examine the fine structure of $C_1(\mathbf{x}_{u_s}, \ell)$. Instead of evaluating Equation 3.1 directly, we record the number of points whose separation from $\mathbf{x}_{u_s}$ is between ℓ_i and $\ell_i + \Delta\ell$ (similar to the method applied to the correlation integral in Chapter 2). A typical calculation uses 100,000 bins to record these separations. The Cantor function $C_1(\mathbf{x}, \ell_j)$ is calculated as the normalized sum over bins 0 to j .

We estimate μ from the statistics of $\hat{\mu}(\ell)$,

$$\hat{\mu}(\ell_j) = \frac{\log C_1(\mathbf{x}, s\ell_j) - \log C_1(\mathbf{x}, \ell_j)}{\log s} \quad (3.15)$$

where s is the period of the oscillation. In practice, the value of ℓ_j is chosen aligned with the bins and the value of $C_1(\mathbf{x}, s\ell_j)$ is interpolated (on a logarithmic scale, the bin density is much greater at this separation). The calculation is repeated for a number of different ℓ_j 's (typically several thousand) spanning one period of the oscillation. The error in μ is determined from the scatter of these estimates; $\hat{\mu}(\ell)$ for the Hénon case is shown in Figure 3.11,

along with the oscillation. Each tick mark on the vertical $\hat{\mu}$ scale corresponds to one standard deviation of the distribution. For the Hénon attractor about x_{us} we find

$$\mu = 1.347 \pm 0.003$$

where the uncertainty corresponds to one standard deviation of $\hat{\mu}$. It is evident from Figure 3.11 that the scatter is biased in sign and concentrated in the fine structure regions of the oscillation -- this is a direct result of the inner cutoff. $\hat{\mu}(\ell)$ shows a self-similar structure which may be taken into account. The bias of $\hat{\mu}(\ell)$ is even more pronounced in the Feigenbaum set (see Figure 3.11'). Here we have

$$\mu(x_0) = 0.75511 \pm 0.002 \quad .$$

If in place of the mean we consider the distribution of the first 10% of the data (in the flat region of $\hat{\mu}$), we have

$$\mu(x_0) = 0.75551 \pm 0.00001 \quad .$$

Returning to the Hénon case and interpreting Equation 3.6 with $d \rightarrow \mu$ and $\sigma \rightarrow \lambda$ yields an effective filling factor ρ of

$$\rho = 2^{\mu \log \lambda_1} \approx 0.082 \quad .$$

This means we expect each blowup in Figure 3.8 to contain a twelve-fold increase in the number of points (corresponding to $1/\rho$). The observed ratios are $r_{ab} = 12.40$; $r_{bc} = 12.89$; and $r_{cd} = 12.22$ where the subscripts denote the two frames considered.

The Cantor function evaluated about an unstable period 2 point is shown in Figure 3.12. The periodic structure is evident. Similar structure has been observed about higher order periodic points.

How is μ related to the more popular scaling exponents? First, μ

provides a bound on d_∞ . Specifically

$$d_\infty \geq \max (\mu(x)) . \quad (3.16)$$

This inequality follows from the sensitivity of d_q to the distribution on the object. d_∞ reflects the scaling structure of the least dense singularities of the set; each value of μ reflects the properties of one (or more) members of the set, thus relation 3.16 follows.

If the singular spectrum is dependent on the periodic orbits, then the details of $f(\alpha)$ may vary when evaluated on different Poincaré sections. The number of intersections that a given unstable periodic orbit makes with a surface of section varies with the surface taken and since $f(\alpha)$ is sensitive to the distribution of singularities, details of $f(\alpha)$ for a given attractor may vary from section to section. It is shown in Chapter 4 that $f(\alpha)$ of a self-similar set may be altered by projection. This is true even in the case that the topological dimensions of the two Euclidean spaces (the original embedding space and the space into which the projection is made) are greater than the Hausdorff dimension of the set. If $f(\alpha)$ is to provide a useful characterization of a system it may be important that the full embedding space be used, or the section chosen with care.

Section 3.5: Conclusions

To conclude we recall the relation of $C_1(\mathbf{x}, \ell)$ to $C_2(\ell)$. For a given set $\mathbf{X}$, consisting of N points,

$$C_2(\ell) = \frac{1}{N} \sum_{i=1}^N C_1(\mathbf{x}_i, \ell) \quad \mathbf{x}_i \in \mathbf{X} \quad (3.17)$$

this implies

$$A(\ell)\ell^\nu \propto \sum_{i=1}^N \alpha_i(\ell) \ell^{\mu_i} \quad (3.18)$$

The point is that, since the sum of power laws is not, in general, a power law, the α_i contain information about the scaling. If the global scaling at small ℓ is dominated by the set of unstable periodic points, the number of terms in the sum is reduced.

The relative strengths of the α_i cannot be fixed, otherwise $\nu = \max(\mu_i) \approx d_\infty$ for ℓ sufficiently small. This requires that the dynamics renormalize the α_i as additional points are considered so that Equation 3.18 holds. We are currently studying the manner in which this balance is maintained.

FIGURE CAPTIONS

Figure	Caption
3.1	The Cantor function of the 101 set about $x=0$. Note the linear coordinate axes.
3.2	The Cantor functions on a log-log scale for the (a) 101 set at $x=0$, (b) $\frac{\log 4}{\log 3}$ Koch curve, and (c) Hénon attractor about the unstable fixed point x_{us}.
3.3	$C_1(x, \ell)$ for the 101 set evaluated at several x values.
3.4	Bifurcation tree for logistic map.
3.5	An approximation of the Feigenbaum set generated by recording the first 2^{13} iterations of the point $x = \frac{1}{2}$ under the logistic map with $\lambda = \lambda_\infty$.
3.6	(a) $C_1(\frac{1}{2}, \ell)$ for the approximation to the Feigenbaum set (see Figure 3.5) and (b) $\psi(x_0, \ell)$ for this set.
3.7	(a) $C_1(x_1, \ell)$ for the approximation to the Feigenbaum set (see Figure 3.5) and (b) $\psi(x_1, \ell)$ for this set.
3.8	The structure of the Hénon attractor near x_{us} . Each panel is a close up of the preceding panel. Each panel is centered on the unstable fixed point and the scale of each panel is a factor of λ larger than the preceding.
3.9	The local correlation integral, $C_1(x_{us}, \ell)$, evaluated for a 2^{24} point approximation of the Hénon attractor.
3.10	The first approximation of $\psi(x_{us}, \ell)$ determined by removing the linear trend computed from $C_1(x_{us}, \ell)$ first approximation.
3.11	(a) One cycle of $\psi(x_{us}, \ell)$ for the Hénon mapping. (b) The matching value of $\hat{\mu}(\ell)$, the ℓ axis is displaced from (a) by one cycle.

- 3.11' (a) One cycle of $\psi(x_0, \ell)$ for the Feigenbaum set
 (b) The matching value of $\hat{\mu}(\ell)$, the ℓ axis
 is displaced from (a) by one cycle.
- 3.12 $C_1(x_{\text{period } 2}, \ell)$ for the Hénon mapping

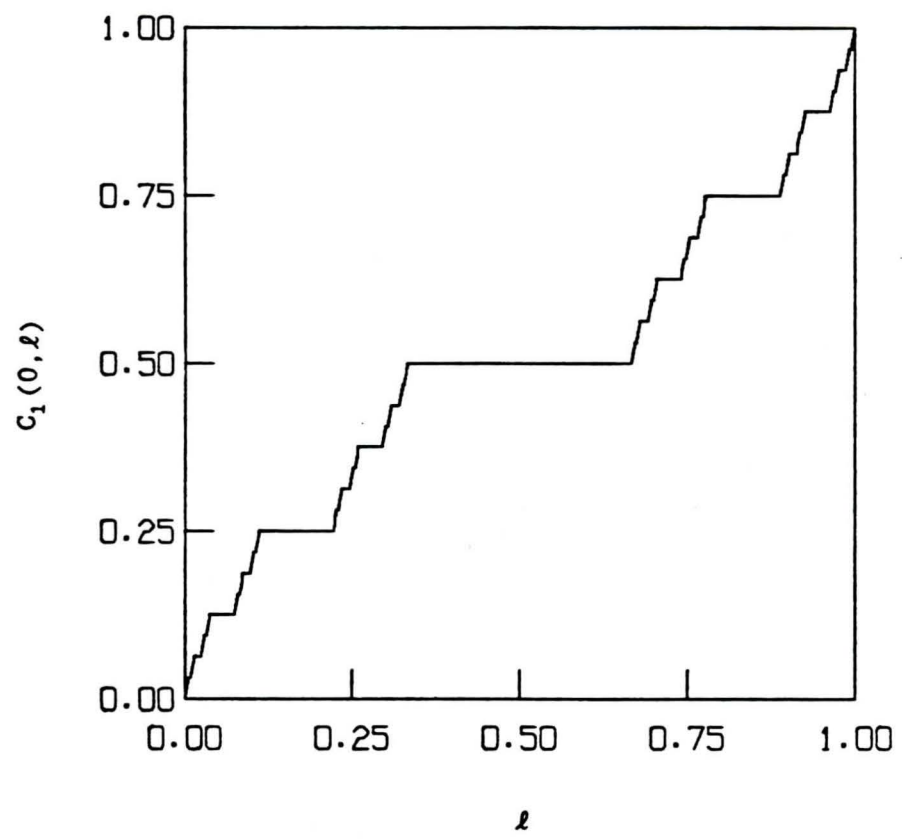


Figure 1

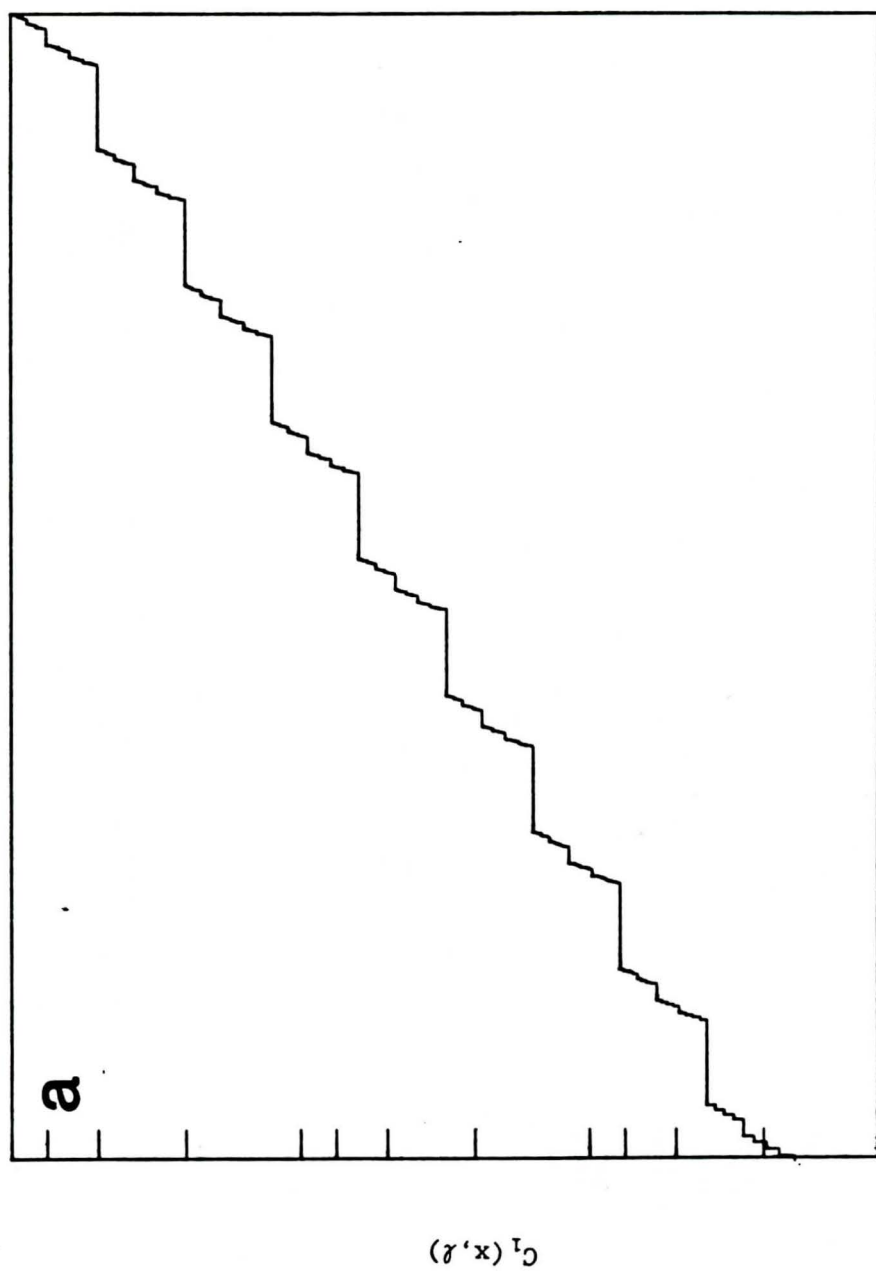
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Figure 2

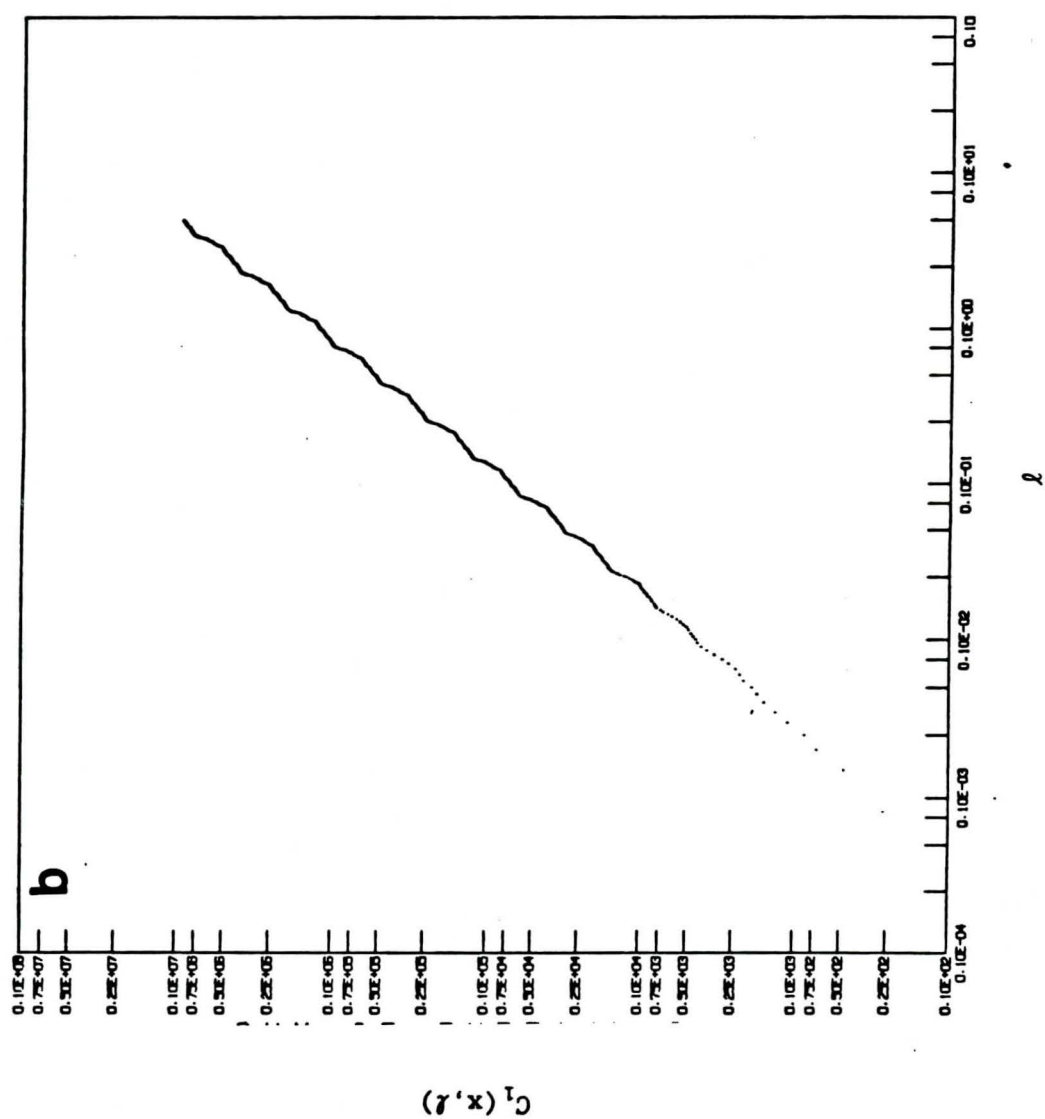


Figure 2

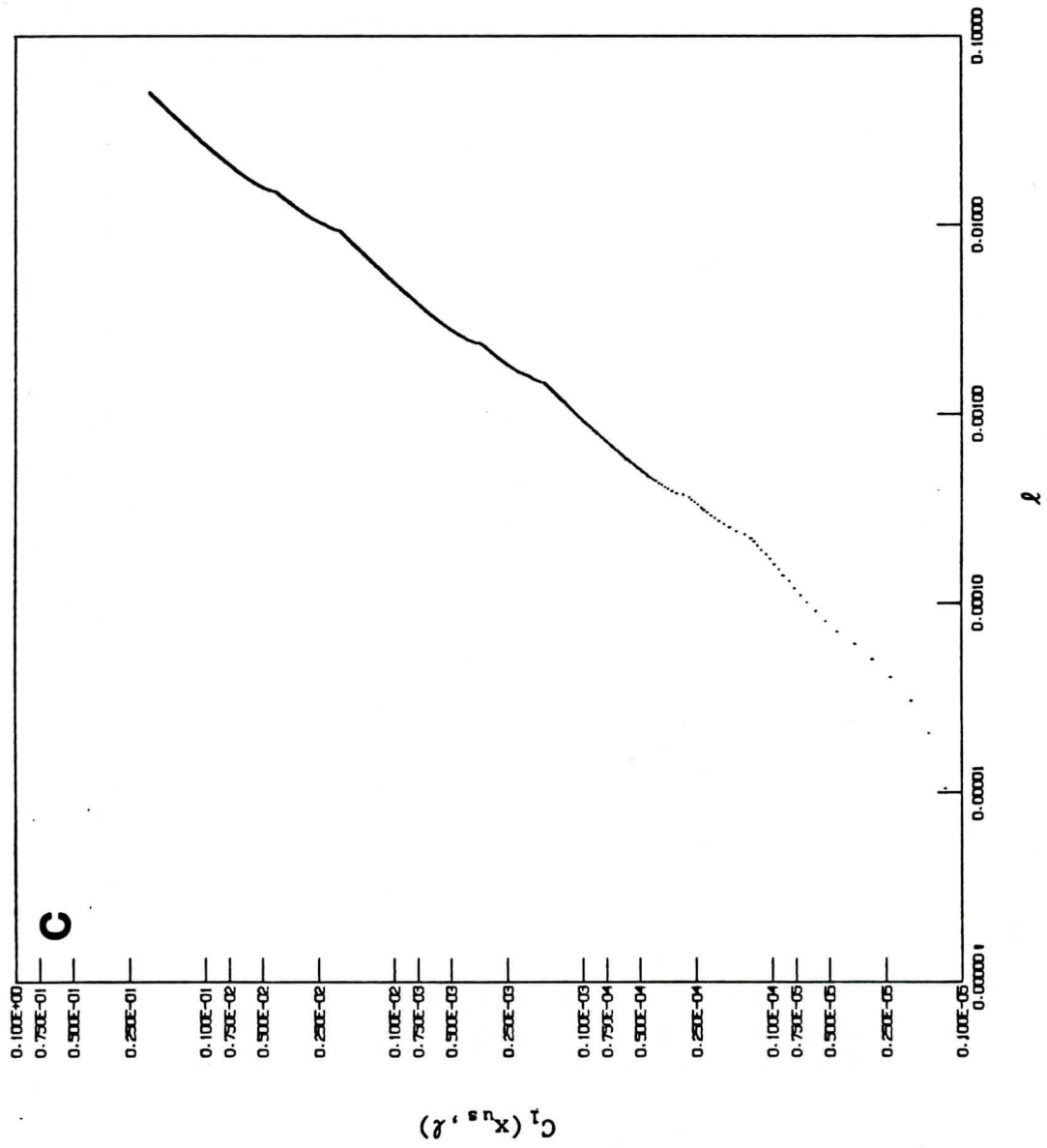


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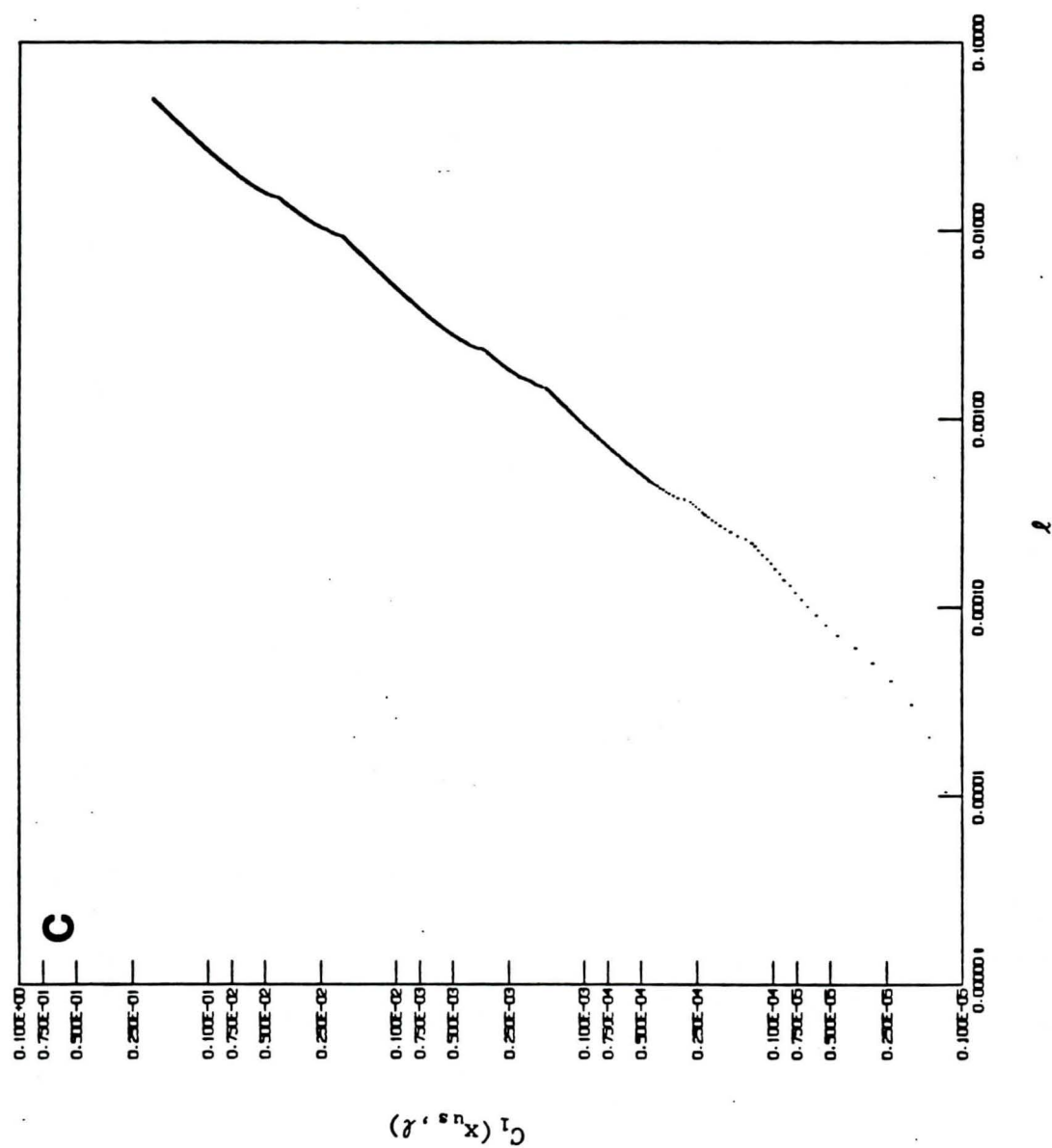


Figure 2

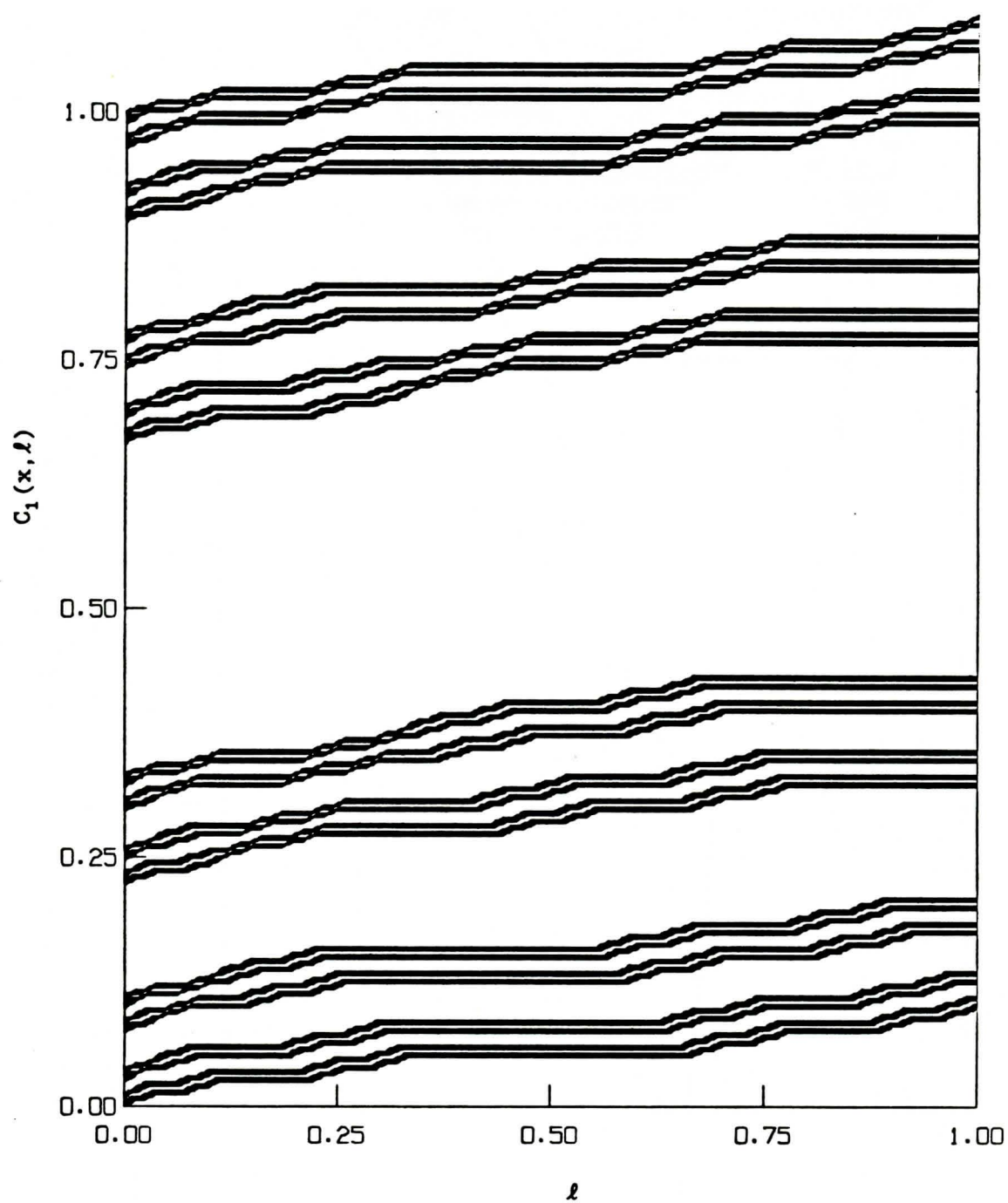


Figure 3

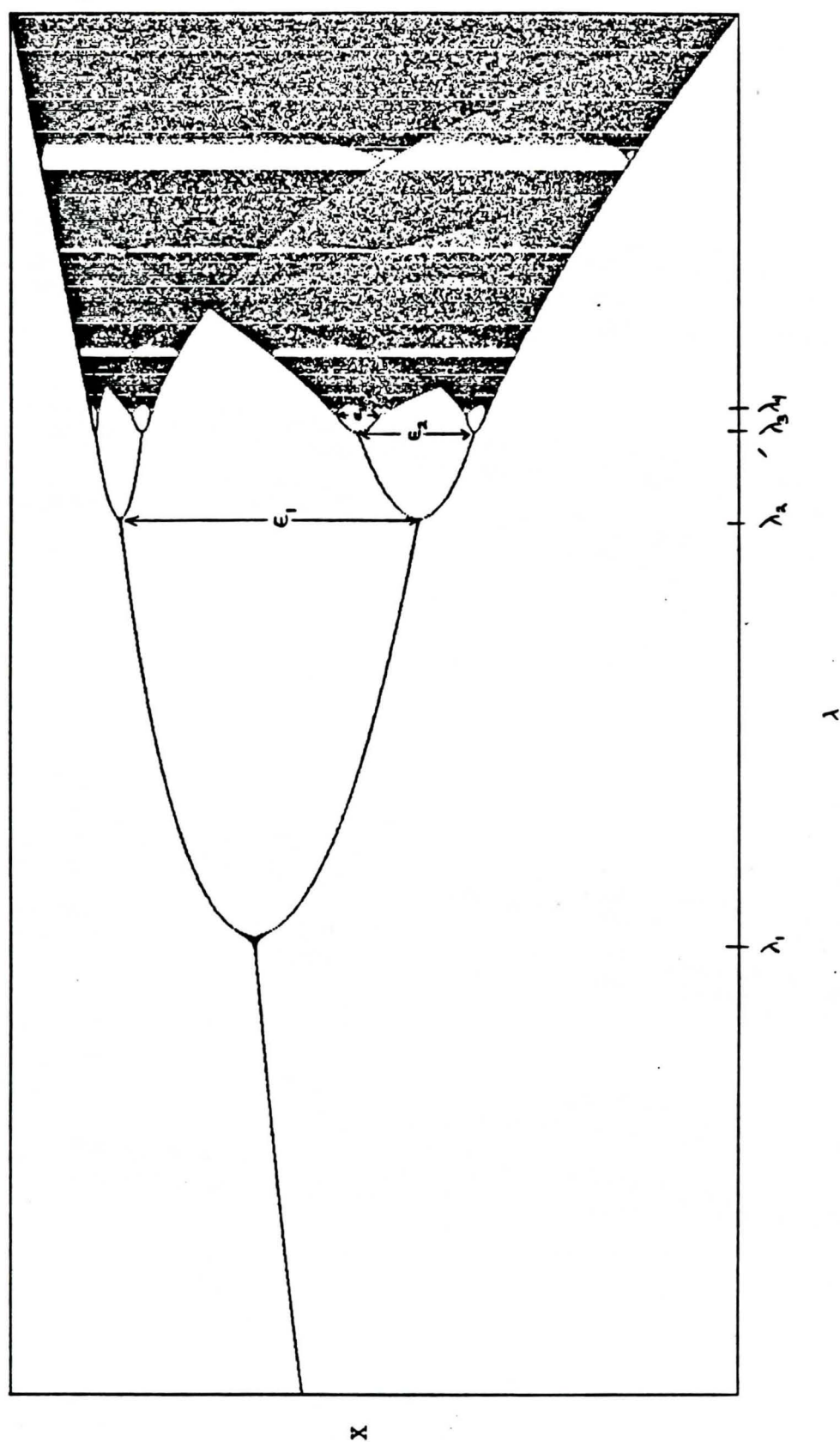


Figure 4

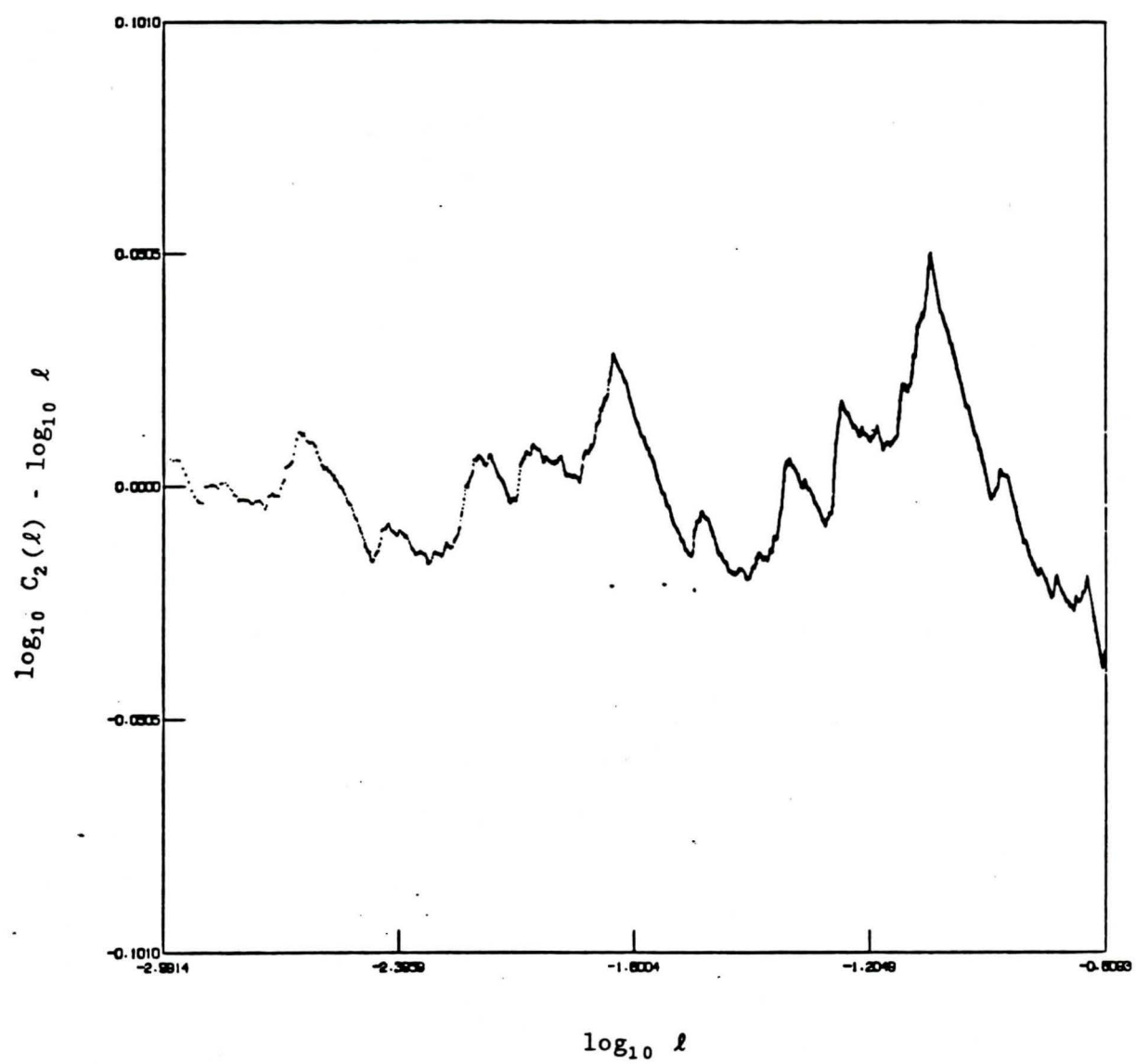


Figure 5 a

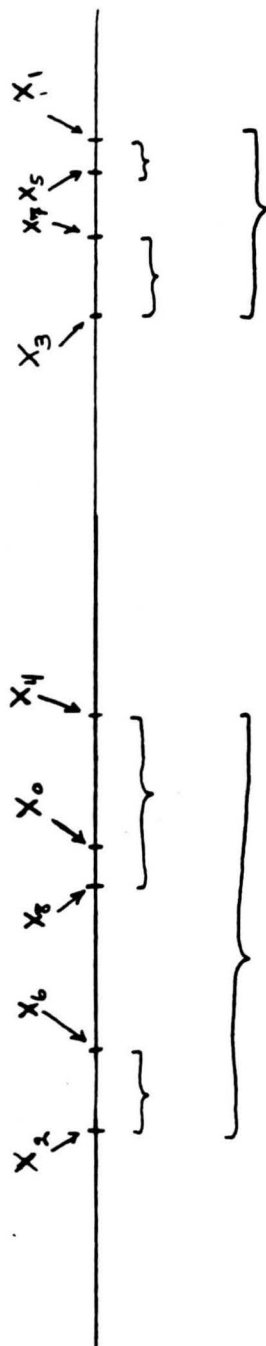


Figure 5 b

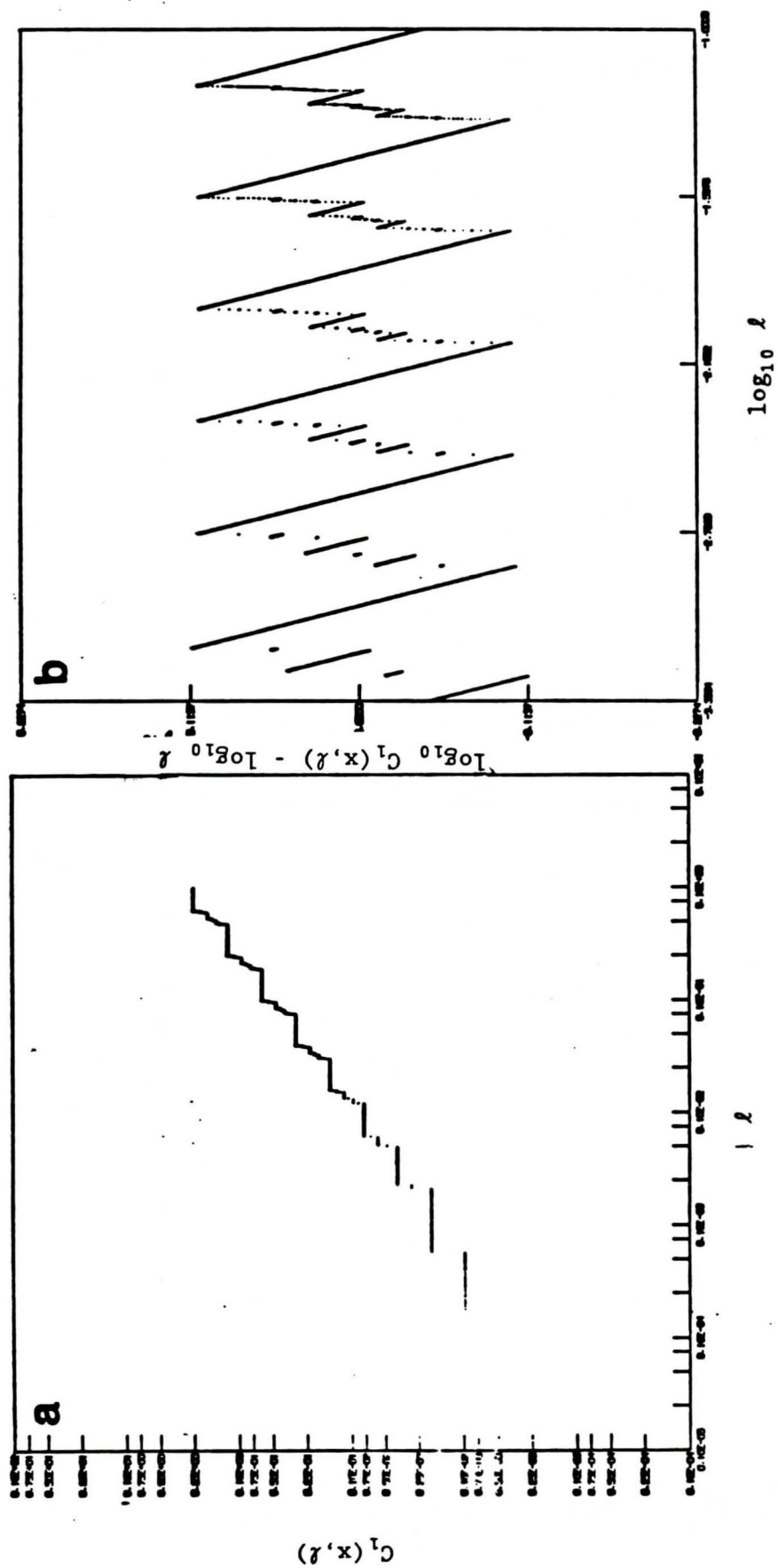


Figure 6

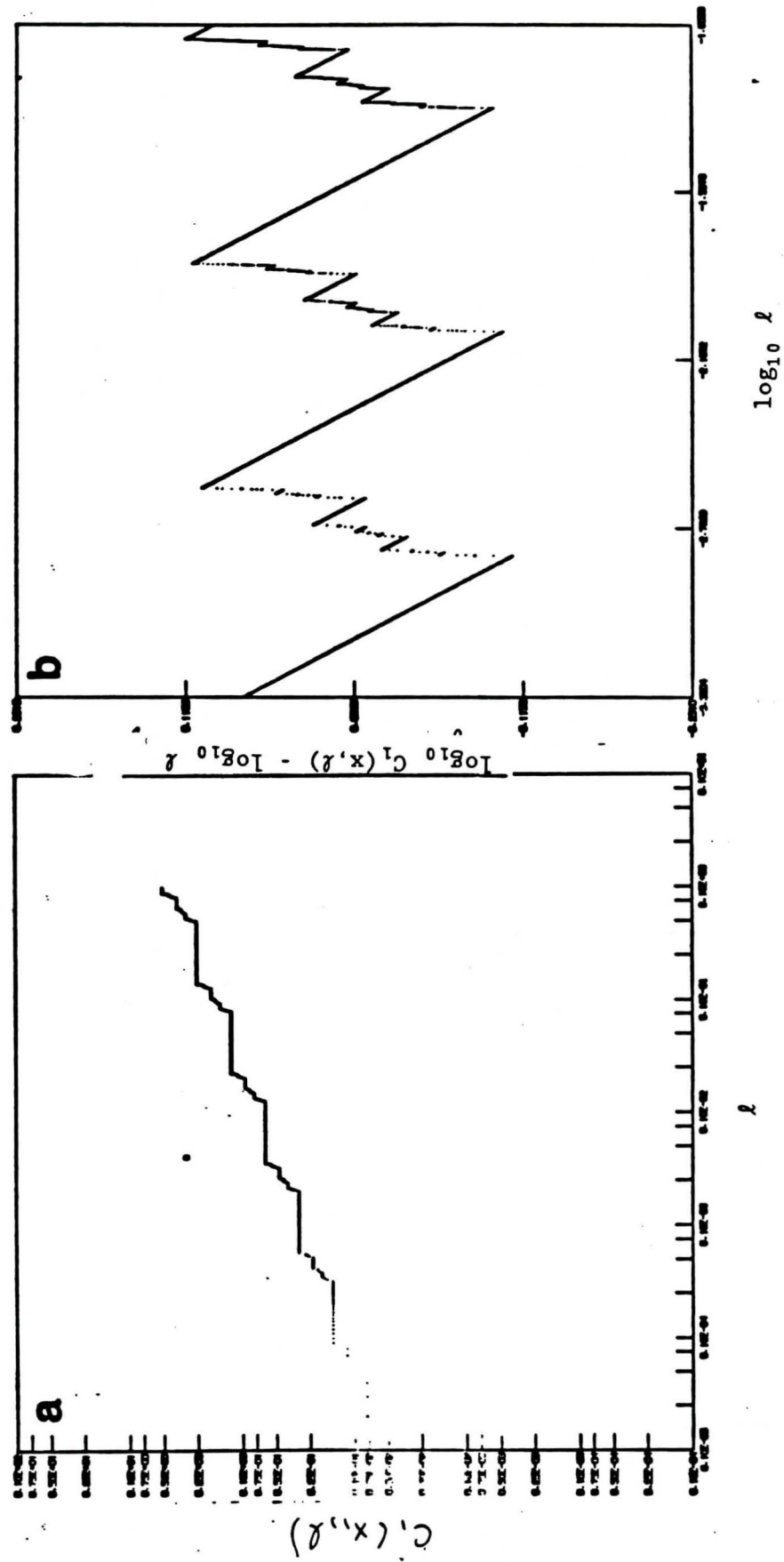


Figure 7

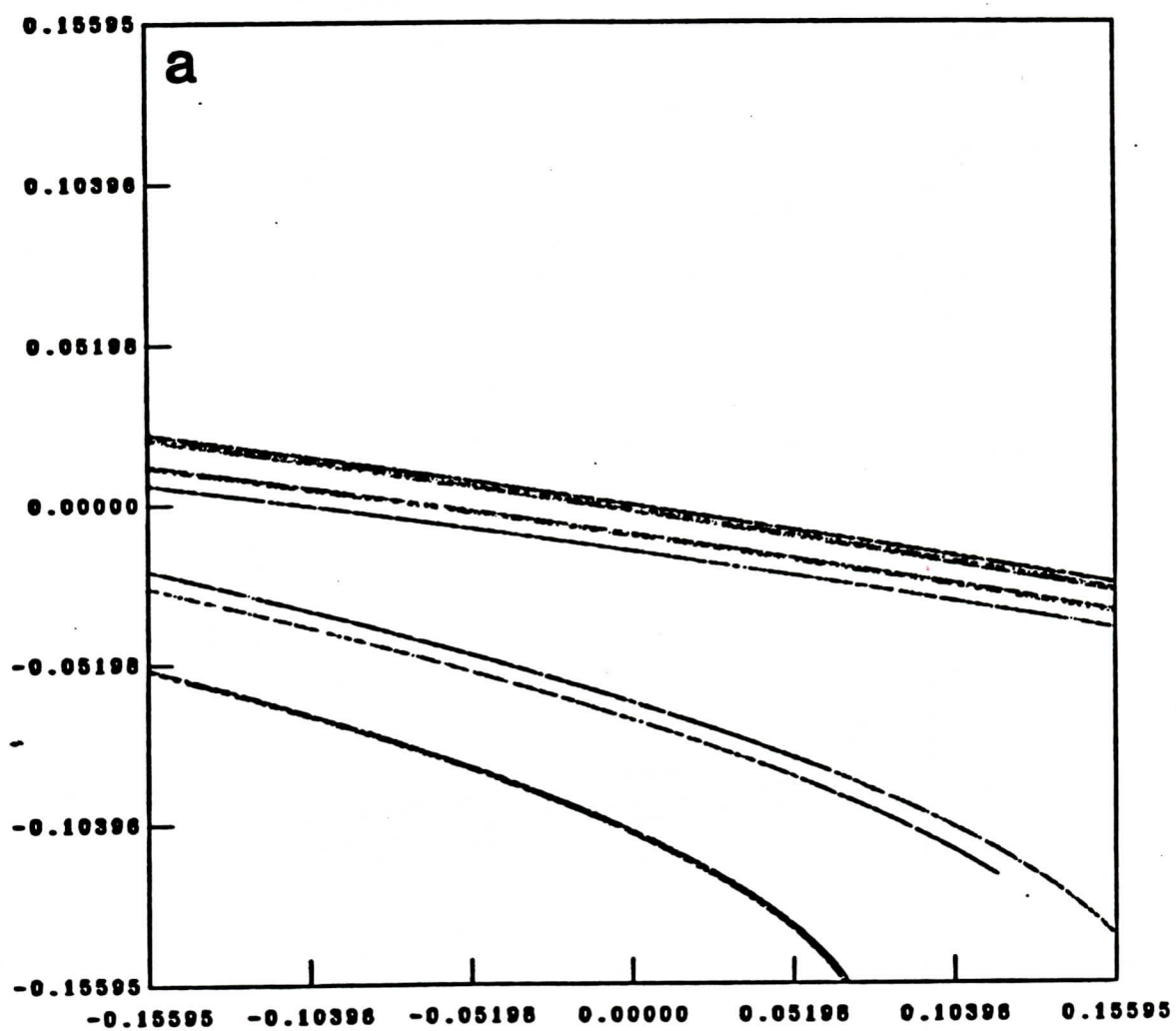


Figure 8

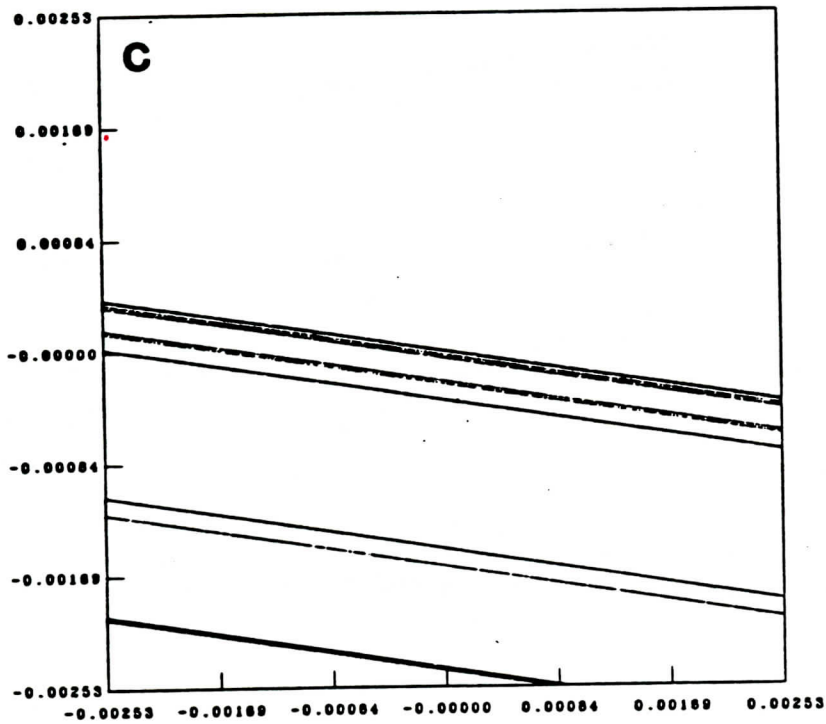
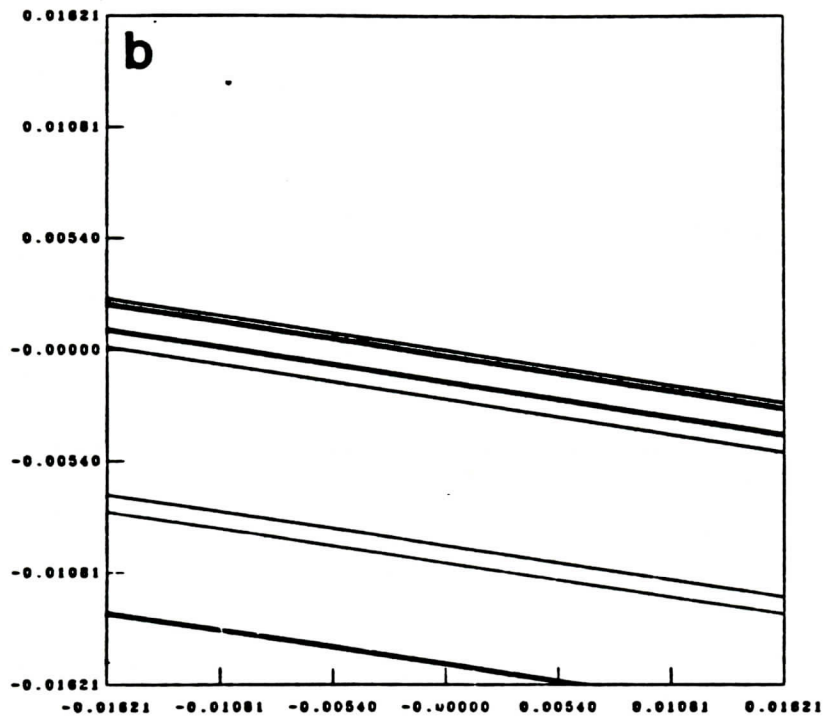


Figure 8

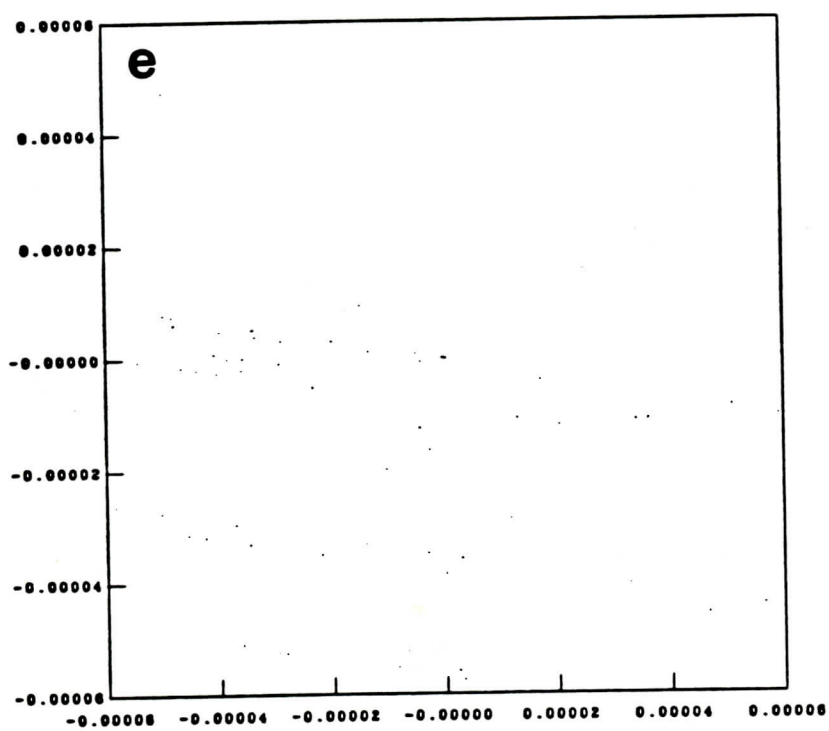
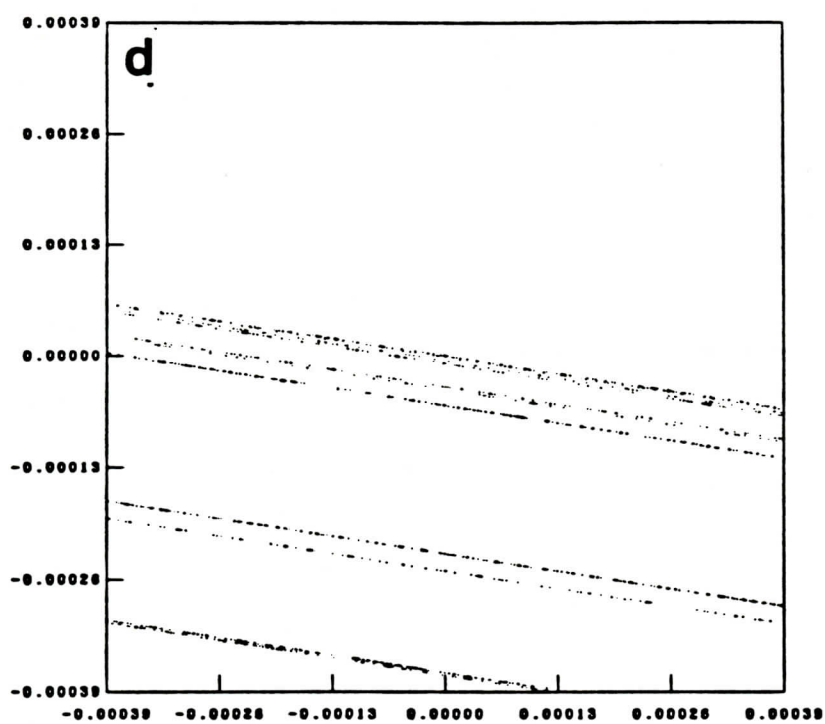


Figure 8

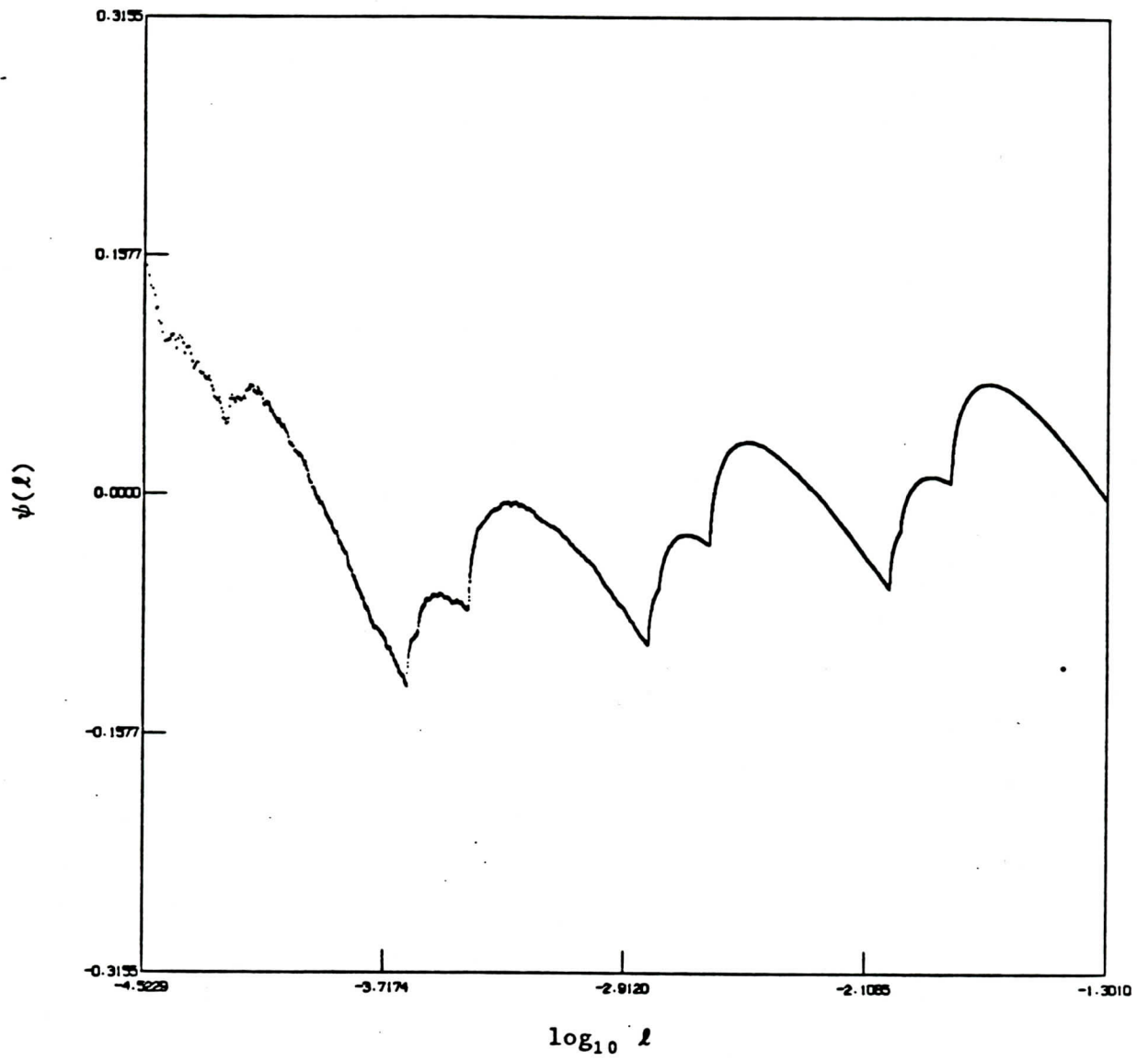


Figure 10

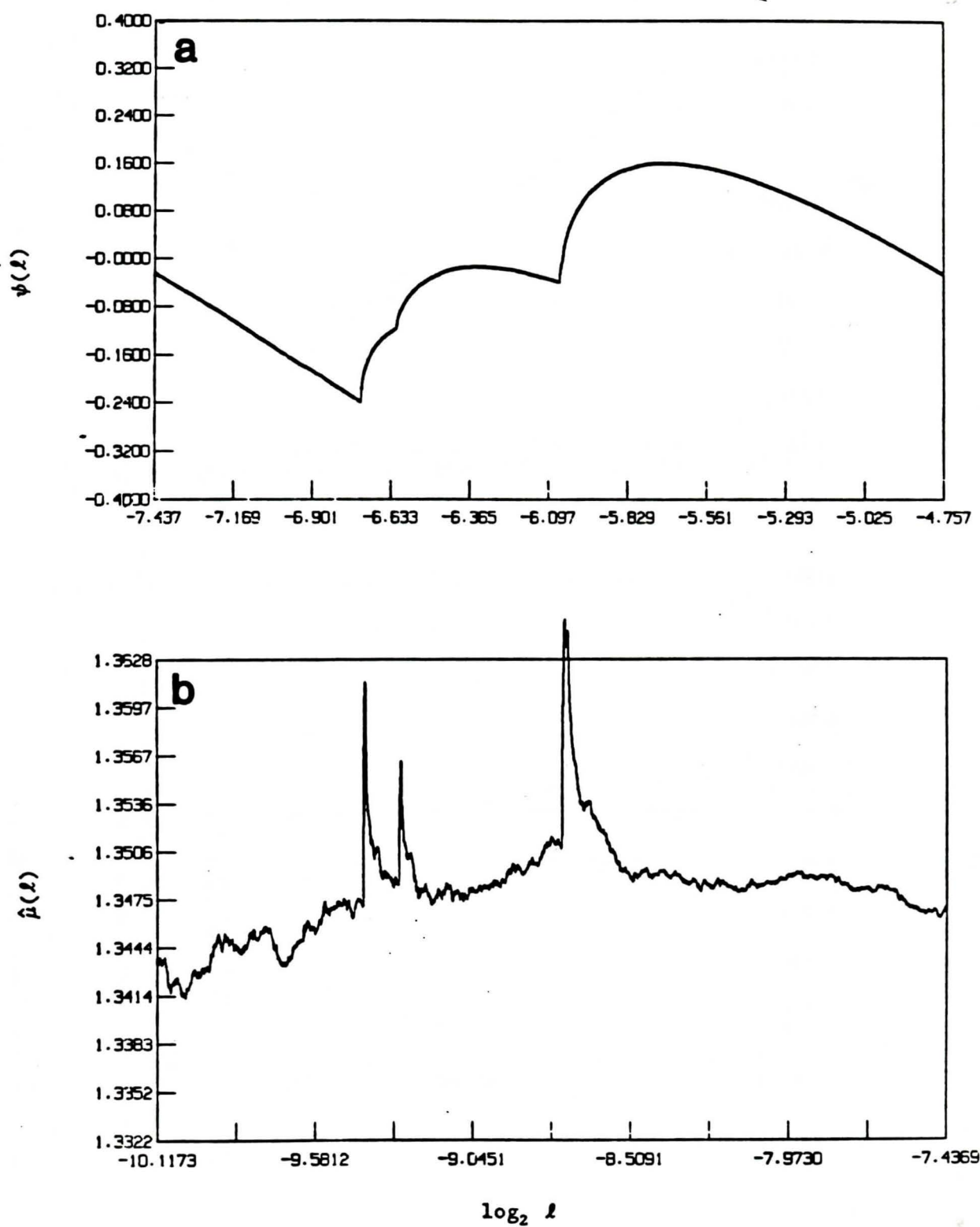


Figure 11

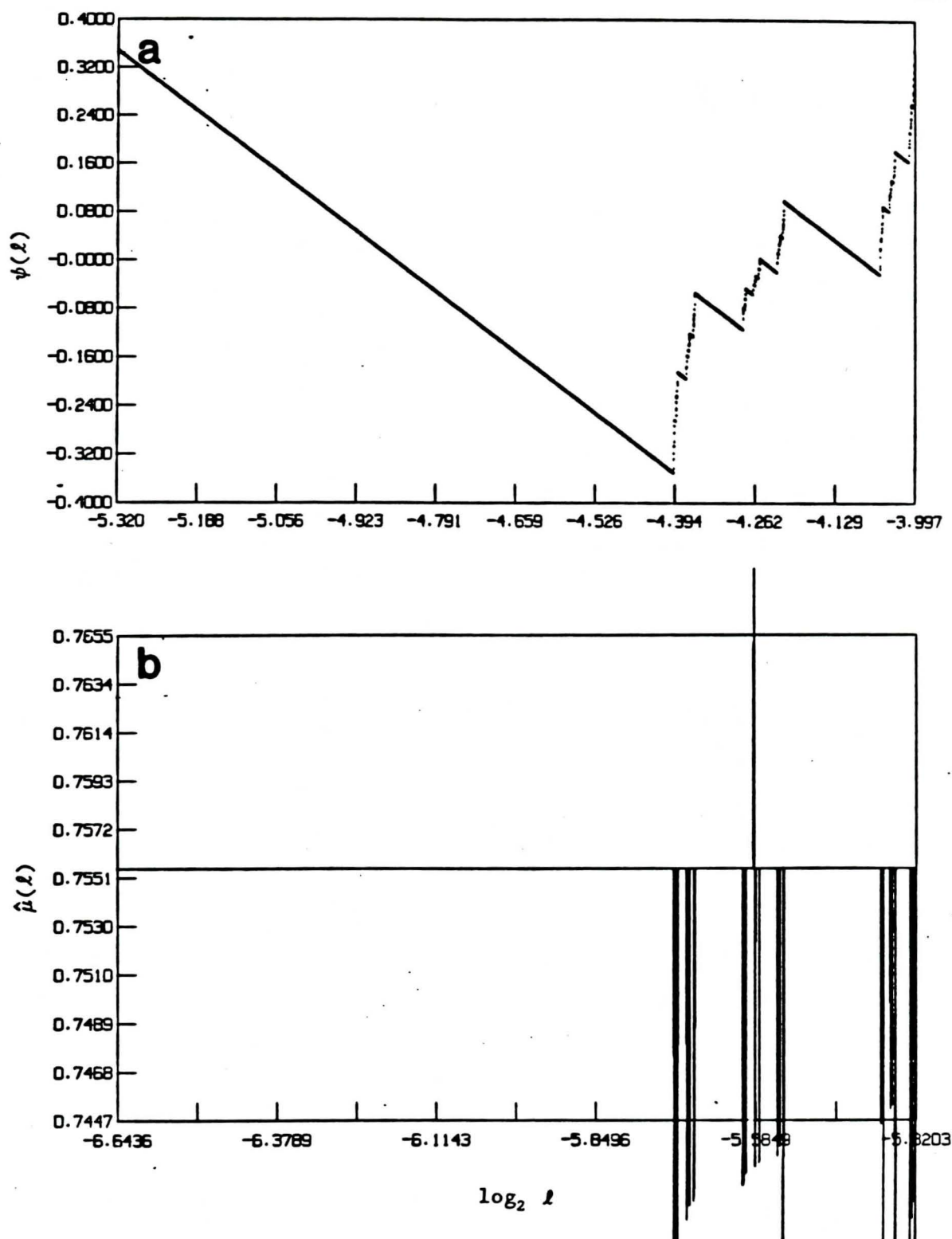


Figure 11'

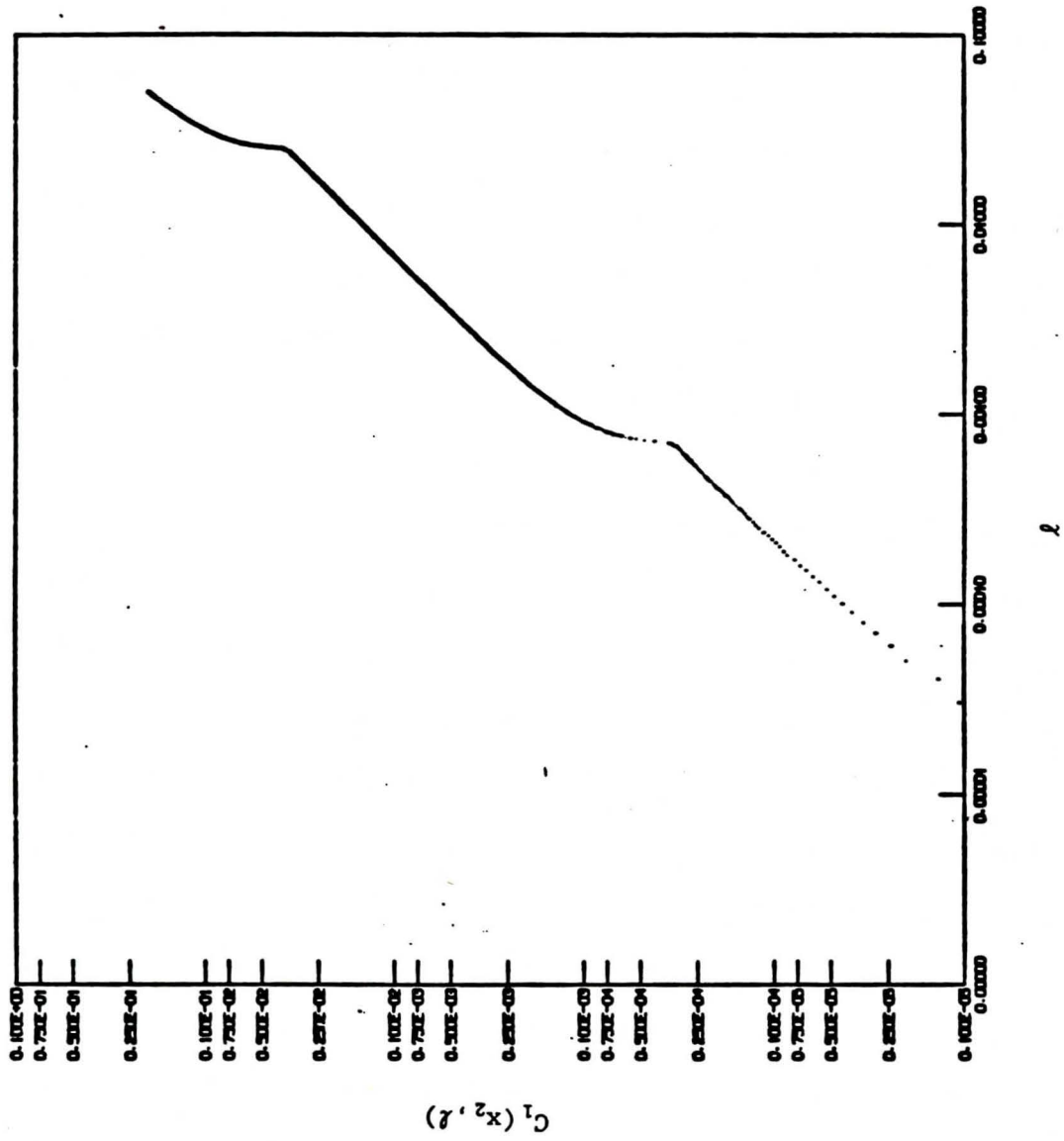


Figure 12

CHAPTER IV Lacunarity in Nature

The concept of self-similarity has received widespread use in a variety of disciplines. Inherently self-similar models have been constructed for the investigation of cosmology (e.g. Fournier, 1907), diffusion limited aggregation models of dielectric breakdown (Sander, 1987), and cascade theories of fluid turbulence (Novikov, 1966; Mandelbrot, 1974). The intrinsic oscillations in the scaling of fractal structures which appear in many of these models will be useful in making the comparison of theory with experiment. In this chapter we will investigate the lacunarity oscillations in the first and third of the examples mentioned above.

Turbulence has been called "the unsolved problem of classical physics" (Cvitanović, 1984). Here we are concerned with a subclass of this problem: the description of three-dimensional, fully developed fluid turbulence. The meaning of these adjectives is made clear below. In the first part of this chapter we demonstrate the role of the oscillation in the β model of fully developed fluid turbulence due to Frisch, Sulem, and Nelkin (1978). We show, via the β model, that oscillations in the scaling statistics are expected as a consequence of the fractal nature of turbulence; predict how the ψ functions of different moments of the velocity distribution are related; and interpret a new scaling parameter for turbulence. Several reasons motivate our choice of the β model. The derivation of the β model is straightforward and the origin of the oscillatory ψ function fits neatly into its development (Smith *et al.*, 1986). Secondly, recent observations by Anselmet *et al.* (1984) are

compatible with and even suggest the existence of an oscillation of reasonably large amplitude in the velocity structure functions of turbulent flows. Finally, other investigators have interpreted the change of slope of the statistical moment versus the logarithm of length scale as an indication of multi-fractal behavior. We do not believe this is always a correct interpretation and demonstrate a characteristic which may distinguish multi-fractal scaling from scaling with an oscillation.

The first part of this chapter is organized into four sections. Section 4.1 contains a brief review of the terminology and phenomenology of the study of turbulence. In Section 4.2 the standard derivation of the β model is presented. Section 4.3 provides a generalized derivation in which the oscillation is seen to arise naturally, notes the favorable comparison with experimental results, and contrasts the behavior of statistics on a multi-fractal and a fractal.

The second portion of this chapter contains an investigation of scaling arguments in cosmology. The distribution of visible matter is non-uniform at scales up to at least 50 megaparsecs (Lapparent, 1986). Evidence for the nested behavior of this distribution (at least at small scales) is stronger than in the case of turbulence. In Section 4.5, the Fournier universe is used to show the implications of viewing a nested scaling structure in projection. Dynamical considerations are noted through the Kida equation and a new generator of such distributions is discussed in Section 4.5. After noting the effects of projection in Section 4.6, an analysis

of current observations is presented in Section 4.7.

Part I Lacunarity in Fluid Turbulence

Section 4.1: Characteristics of Turbulence

Richardson (1922) was the first to propose that a turbulent flow field could be envisioned as a collection of eddies of all sizes. Richardson's observation was that an active region of size ℓ is spatially composed of several smaller more intensely active regions. The distribution of the largest structures is neither spatially nor temporally uniform. Hot wire anemometer measurements of turbulent velocity fields by Batchelor and Townsend (1947) quantified this behavior which they termed "intermittency." A typical velocity trace recorded at a fixed position in a turbulent flow reveals long periods of quiescence interrupted by irregularly spaced bursts of activity; this behavior is the distinguishing characteristic of intermittency in turbulence. Intermittency is a property of small scales; active bursts are not space-filling. The behavior of a temporal trace is related to spatial intermittency through to the mean advection of this flow: it is assumed that the active regions that are carried past the fixed detector provide a good representation of the flow's characteristics. Landau (1944) suggested that the transition to turbulence could be understood as the activation of more and more Fourier modes; while this model of the transition to turbulence is not universally excepted, the swirls visualization has become the textbook model of fully developed turbulence (Landau and Lifshitz, 1979 §31).

Consider a fluid in turbulent motion: if the flow is to maintain itself, the constant dissipation of energy by viscosity at the smallest active scales, must be balanced by an inflow of energy. It is often the case that this source introduces energy into the system at large scales. In a turbulent energy spectrum (see e.g. Tenekes and Lumley, 1972) a range of wave numbers over which the energy scales as the $5/3$ power of the wave number is often observed. This was predicted by Kolmogorov (1941) on the basis of dimensional arguments. This inertial range corresponds to length scales ℓ such that

$$\ell_0 \gg \ell \gg \ell_d \quad (4.1)$$

where ℓ_0 is the scale at which energy is introduced into the system; ℓ_d is the scale at which viscous damping becomes important. As emphasized by Frisch and Morf (1981), the scales below ℓ_d do not constitute a strictly damping region; both damping and inertial effects play a part at these scales.

The goal is to follow the path of the energy introduced into the system as it cascades to smaller length scales (larger wave numbers) and is eventually dissipated by viscosity. We will frame our picture in wave number space (k-space) and consider the energy contained in "eddies of size ℓ ", or equivalently, the energy in a thin spherical shell of k space with radius $k = 2\pi/\ell$. Kolmogorov assumed the energy cascade was local in this space, passing from one shell to the next until the dissipative range was reached at large k . This assumption is often considered equivalent to saying that energy is transferred only between eddies of similar sizes. Note that in this picture an

eddy is something other than a blob of swirling fluid with a characteristic size ℓ . There are special cases in which the swirling blob picture may be shown to have its energy concentrated in a fairly narrow band of k values near ℓ^{-1} (see, e.g. Beran (1968), 344); Townsend (1956) relates the spatial structure to the k spectrum for several simple distributions. Nonetheless, the energy in eddies of size ℓ is the energy in a region of k space defined by an integral over the entire spatial domain.

Turbulence is considered fully developed once a dynamic equilibrium is established and the energy dissipated by viscous forces, E , in some characteristic time of the system, t , is equal to the rate at which it is introduced into the system. Denoting this quantity by ϵ , we have

$$\epsilon \equiv \frac{E}{t} = \nu \langle (\nabla v)^2 \rangle \quad (4.2)$$

where $\langle \dots \rangle$ denotes ensemble average and ν is the viscosity of the fluid.

Scaling behavior in fluid motion is not unexpected. It is easily seen (eg. Benzi et al., 1984 or Frisch, 1983) that the Navier-Stokes equations for the velocity fields $v(x)$ of a viscous fluid (with to no external force)

$$\partial_t v(x) + (v(x) \cdot \nabla) v(x) = -\nabla p / \rho + \nu \nabla^2 v(x) \quad (4.3)$$

are formally invariant under the scale transformations

$$r \rightarrow \sigma r, \quad v \rightarrow \sigma^h v, \quad t \rightarrow \sigma^{1-h} t, \quad \nu \rightarrow \sigma^{h+1} \nu; \quad \sigma > 0. \quad (4.4)$$

For arbitrary values of h . In the above ρ is the density and p the pressure.

Under this transformation, the energy dissipation scales as is

$$\epsilon \rightarrow \sigma^{3h-1} \epsilon$$

With the assumption that the relevant nonlinear interactions are local in k space, the invariance of the energy dissipation

$$\sigma^{3h-1} \epsilon = \epsilon \Rightarrow h=1/3. \quad (4.5)$$

In this case,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x^h} = \lim_{\Delta x \rightarrow 0} \frac{v(x) - v(x+\Delta x)}{(\Delta x)^{1/3}} = 1 \quad (4.6)$$

and, therefore, the velocity gradient is singular. Kolmogorov (1941) originally assumed that the set of singular points was space filling. In an attempt to understand the spatial intermittency of turbulence, Landau (1944) pointed out that if the set of singular points had a dimension less than three, the energy dissipation would be a fluctuating quantity in space. Novikov and Stewart (1964) provided the first self-similar mathematical model of the energy cascade. These ideas were extended by Kraichnan and culminated in the β model of Frisch, Sulem and Nelkin (1978).

The difference between turbulent and laminar flow is often visually apparent. The complexity of a turbulent field suggests a statistical description, such as the energy spectrum. Some experiments seek to provide a detailed history of individual fluid elements. One approach is to follow the path of a "passive tracer" suspended in the fluid and reconstruct the fluid motion from that of the tracer. The composition of the tracer may vary considerably; Richardson and Stommel (1948) used parsnips to follow eddy motion on the surface of a river, while a line or sheet of hydrogen bubbles is

often used to trace motions within a fluid system. In turbulent flows, fluid lines are expected to grow exponentially (Corrsin, 1950; see also Cocke, 1969). Sreenivassen *et al.* (1986) follow a line of hydrogen bubbles in a turbulent shear flow, measuring its length as it is advected downstream. The line is stretched and folded by the flow, and fractal structures are observed to develop. The task is then to relate the fractal distributions of the tracer to the motion of the host fluid. In Chapter 5, it is shown that while exponential growth of a material line of a tracer is a necessary condition for turbulence, such observations are not sufficient to infer turbulent flow. It is demonstrated that such tracer distributions may develop very complex, self-similar structure in very simple, non-turbulent flows; exponential growth of a material line does not imply turbulence. The development of this self-similar structure has been recently observed in a surface flow by Chaiken *et al.* (1986, 1987).

When describing the velocity field, the moments of the velocity distribution (called the velocity structure functions) provide a common vehicle for the presentation of experimental results and their comparison with theory. In a homogeneous, isotropic flow, the p^{th} order velocity structure function is defined as

$$S_p(l) \propto \langle (\delta v(l))^p \rangle = \langle (v(\mathbf{r} + \mathbf{l}) - v(\mathbf{r}))^p \rangle \quad (4.7)$$

where $\mathbf{r}$ represents a point in the flow and $\mathbf{l}$ defines a separation vector. For isotropic turbulence, all orientations of $\mathbf{r}$ and $\mathbf{l}$ are considered and the average is taken over physical space. For a jet

flow (e.g. Anselmet *et al.*, 1984) only the component of the velocity parallel to the mean flow is considered and the displacement is likewise taken only in this direction. High order structure functions are often normalized by the second order moment, forming the nondimensionalized structure functions,

$$a_p(\ell) = \frac{S_p(\ell)}{[S_2(\ell)]^{p/2}} \quad (4.8)$$

In the inertial range, the structure functions are observed to scale as a power of ℓ . Differences in the relationship between exponents of different moments provide a method to distinguish between models experimentally. In Kolmogorov's 1941 theory, the structure functions are independent of p in the inertial range. As shown below, the β model implies a linear dependence on p while the lognormal model (Kolmogorov, 1962) has a quadratic dependence. Anselmet *et al.*'s experiments fall between the last two, but in the high order moments, the dependence appears to depart from linearity.

The β model has spawned a host of more realistic progeny including the α model (Lovejoy, private communication) and the random β model (eg. Benzi *et al.*, 1984). The results of our own generalization may be straightforwardly (if not easily) applied to the more modern versions. The main point of this chapter is that in addition to the mean scaling, the fractal nature of turbulence implies an oscillatory behavior in the $S_p(\ln(\ell))$. This result was foreseen by Novikov (1964).

Section 4.2 : The β Model

The β model considers the energy cascade of a three dimensional flow. Consider starting with an active region at the outer length scale, ℓ_0 . The region is divided into λ^3 smaller subregions representing eddies "nearby" in k-space. The original active eddy feeds its energy into N active eddies of size $\ell_1 \approx \ell_0/\lambda$. Figure 4.1 attempts to show this schematically. After n generations, there are N_n active 'n-eddies' of scale ℓ_n where

$$\ell_n = \lambda^{-n} \ell_0 \quad (4.9)$$

Frisch *et al.* (1978) take λ equal to 2, our departure from their notation stems from a desire to track the role of this parameter. Kolmogorov's space-filling model corresponds to the case where all λ^{3n} n-eddies are active. In the β model, N is less than λ^3 – not all of the subregions are active – and the cascade continues only within a fraction of the space. In this model, the small scale activity is forced to be nested within larger active regions.

Denoting the active fraction at each step of the cascade as β , we find

$$\beta(\lambda) = (N/\lambda^3) \quad (4.10)$$

In order to compute the structure functions for the β model, it is necessary to determine the typical velocity difference across a region of size ℓ . After n generations, the fraction of the original volume occupied by active eddies is

$$\begin{aligned}\beta_n(\lambda) &= (N/\lambda^3)^n \\ &= \lambda^{-(3-D)n}.\end{aligned}\quad (4.11)$$

These n -eddies will contain an energy per unit mass

$$E_n \approx \beta_n(\lambda) v_n^2 \quad (4.12)$$

where v_n is the typical velocity difference at scale ℓ_n . Taking an energy transfer time, t_n , to be on the order of the eddy turnover time, ℓ_n/v_n , we obtain the energy transfer rate

$$\epsilon_n = E_n/t_n$$

as

$$\epsilon_n \approx \beta_n(\lambda) v_n^3 / \ell_n. \quad (4.13)$$

Imposing equilibrium in the energy transfer rate ($\epsilon_n = \epsilon$ for all n) then yields

$$v_n \approx \epsilon^{1/3} \ell_n^{1/3} \beta_n(\lambda)^{-1/3} \quad (4.14)$$

In Frisch et al (1983), $\lambda = 2$ and β_n takes the form

$$\beta_n(\lambda) = \beta^n, \quad \beta = N/2^3 \leq 1 \quad (4.15)$$

Assuming

$$N = \lambda^D = 2^D,$$

they find

$$v_n \approx \epsilon^{1/3} \ell_n^{1/3} (\ell_n/\ell_0)^{(3-D)/3} \quad (4.16)$$

where D is the Hausdorff dimension of the set of singularities of the velocity field and is related to the number of offspring through

$$D = \log_{\lambda} N = \log_2 N \quad (4.17)$$

A value of D less than three implies the set of singularities is not space filling and hence spatial intermittency.

Now consider the velocity structure function of β model flows. The evaluation Equation 4.7 requires the average velocity difference as a function of space. In the β model, the velocity difference across an active region of scale ℓ_n is simply $v_n(\ell_n)$. Only a fraction $\beta_n(\lambda)$ of the total volume is active; the remainder is inactive and therefore does not contribute to the average. The p^{th} order structure function is

$$\begin{aligned} S_p(\ell_n) &= \langle \delta v_n(\ell_n)^p \rangle \\ &\approx \beta_n(\lambda) \langle \delta v_n(\ell_n)^p \rangle_{\text{active}} \\ &\approx \beta_n(\lambda) v_n(\ell_n)^p \end{aligned} \quad (4.18)$$

where $\langle \dots \rangle_{\text{active}}$ means the average is taken only over active regions. Substituting from Equations 4.15, 4.16 and 4.17, the structure functions are

$$S_p(\ell_n) \approx \epsilon^{p/3} \ell_n^{p/3} \left(\frac{\ell_n}{\ell_o} \right)^{\zeta_p}$$

where

$$\zeta_p = \frac{1}{3} (3-D)(3-p). \quad (4.19)$$

Similarly, the dimensionless structure functions are

$$a_p(\ell_n) \approx \left(\frac{\ell_n}{\ell_o} \right)^{\xi_p}$$

where

$$\xi_p = (3-D)(2-p)/2. \quad (4.20)$$

The slope of the β model structure functions varies linearly with

the order of the moment. This is the result of Frisch *et al.*

Section 4.3: β Model Oscillations

We return now to the velocity equation (4.14) and repeat the analysis retaining λ explicitly. In doing so, we will observe that while the scaling remains independent of λ a generalized similarity ratio is introduced. At this point $\beta_n(\lambda)$ is defined only at $\ell = \ell_n$. We want to transform β into a function for arbitrary ℓ . First recall that for an arbitrary magnification α , the general solution to a scaling for a set of dimension d and similarity ratio σ implies

$$C(\alpha\ell) = \alpha^d C(\ell) \exp\{\psi(\log(\alpha\ell)/\log(\sigma)) - \psi(\log(\ell)/\log(\sigma))\}$$

where $\psi(\log(\ell))$ is a periodic function of period one (for $\alpha = m\sigma$ and integer m we recover $C(\alpha\ell) = \alpha^d C(\ell)$).

Generalizing Equation 4.11 we have

$$\beta_n(\lambda) \rightarrow \beta(\lambda, n) = \lambda^{-n(3-D)} \exp \left\{ f \left[-n \frac{\log(\lambda)}{\log(\mathcal{L})} \right] - f \left[\frac{\log(\ell/\ell_0)}{\log(\mathcal{L})} \right] \right\} \quad (4.21)$$

where $f(\log \ell)$ is a ψ function with period $\log \mathcal{L}$.

Solving Equation 4.9 for n , we find

$$n = - \frac{\log(\ell/\ell_0)}{\log(\lambda)} \quad (4.22)$$

where we have dropped the subscript on ℓ along with the restriction

to integer n . Substitution of this expression into Equation 4.21 yields the desired expression for β as a continuous function of ℓ . Specifically

$$\beta(\ell) = \left(\frac{\ell}{\ell_o} \right)^{-3+D} \exp \left\{ f \left(\frac{\log(\ell/\ell_o)}{\log(\mathcal{L})} \right) - f(0) \right\} \quad (4.23)$$

where λ has cancelled and a new length ratio $\mathcal{L}$ characteristic of the flow has entered. With this form for β , the velocity becomes

$$v(\ell) = \epsilon^{1/3} \ell^{1/3} \left(\frac{\ell}{\ell_o} \right)^{(3-D)/3} \exp \left\{ -\frac{1}{3} \left\{ f \left(\frac{\log(\ell/\ell_o)}{\log(\mathcal{L})} \right) - f(0) \right\} \right\} \quad (4.24)$$

and the oscillatory correction is present in the structure function as

$$S_p(\ell) = \epsilon^{p/3} \ell^{p/3} \left(\frac{\ell}{\ell_o} \right)^{\zeta_p} \exp \left\{ \left(\frac{3-p}{3} \right) \left\{ f \left(\frac{\log(\ell/\ell_o)}{\log(\mathcal{L})} \right) - f(0) \right\} \right\} \quad (4.25)$$

and

$$a_p(\ell) = \left(\frac{\ell}{\ell_o} \right)^{\zeta_p} \exp \left\{ \frac{2-p}{2} \left\{ f \left(\frac{\log(\ell/\ell_o)}{\log(\mathcal{L})} \right) - f(0) \right\} \right\} \quad (4.26)$$

If the intermittent nature of turbulence is due to a self-similar structure, we expect there to exist oscillations about the mean scaling of the velocity structure functions. We reiterate here that these oscillations are a result of the fractal nature of the system, not a particular result of the β particular model used. Observing the oscillation may prove quite difficult; yet such an observation would be very valuable in that it would provide additional parameters in the description of turbulent flows. Since the energy cascade is

not strictly local in k space, one might interpret $\mathcal{L}$ as an indicator of the range over which eddies interact; specifically, the logarithm of the ratio of the outer radius to the inner radius of the k shells which are coupled to the wave number in question.

While Equation 4.25 does not determine the amplitude or period of the oscillation, it does predict a relationship between the oscillations observed in structure functions of various orders; specifically that their periods are equal and their amplitudes determined by the extreme values of ψ . The β model ψ function yields

$$\text{range of } |\psi_p(\ell)| = \left[\frac{2-p}{2} \right] |f_{\max} - f_{\min}| \quad (4.27)$$

In the β model, the magnitude of ψ increases linearly with p . A central assumption here is that the singularities of the set are governed by a single self-similar set, which may be described by a single dimension and one basic ψ function. The comparison with multi-fractals is given below.

Oscillations have been observed in the structure functions of shear jet turbulence in a recent experiment by Anselmet *et al.* (1983). They note "oscillations, which are only weakly manifested for $n = 10$ and 12 , are rapidly amplified at larger n ," where n is the order of the moment. From their Figure 13, we judge the period of oscillation appears to take on the same value for each moment. Independently of Anselmet, van der Water *et al.* (1987) have observed what they believe to be a robust oscillation in the reconstructed attractor from the velocity measurements in a turbulent (air) boundary layer.

The source of these observed oscillations has not been

established. They may result from factors other than those discussed here. Nonetheless, oscillatory corrections are expected if the fractal on which the set of dissipation lives is lacunar, and will assist in determining the characteristics of that set. It has been suggested that the oscillation provides an insight much more deeply coupled to the underlying physics than the mean scaling. Indeed, Novikov (1986) has remarked that linearly scaling models should be classified as "pre-models": such models are wrong. The physics is to be found in understanding the deviations from pure scaling.

The term "multi-fractal" was coined by Frisch and Parisi (1983) to describe a set constructed from a combination of fractal sets each of differing Hausdorff dimension. The relevance of multi-fractals to turbulence arises in the limit of zero viscosity. In this limit, the Navier-Stokes equations (4.3) are invariant under transformations 4.4 for any value of h . Assuming that singularities of different orders do exist, different moments of the velocity structure functions will be sensitive to the distribution of different sets of singularities (see Frisch and Parisi for more details). This would result in a deviation of the slope of S_p vs ℓ with increasing p . Such a decrease is observed by Anselmetti, although it is small compared to the error estimates.

If the scaling of different moments are governed by independent fractal distributions, each may have its own $\mathcal{L}$. In the β model above, only one set is considered and a single dimension and ψ function are predicted. Experimental discrimination of multi-fractal scaling may prove very difficult; individual sets should dominate

only at small separations. If such sets have different dimensions, the arguments above establishing a unique period for all moments should fail; conversely the observation of a single period would then cast doubt on a multi-fractal interpretation.

Can a homogeneous fractal masquerade as a multi-fractal? Yes. If the range of observation covers only a portion of an oscillation of a homogeneous fractal, the variation in ψ may be misinterpreted as a change in the mean slope. The two cases may be distinguished by increasing the range of observations, a difficult task at best.

The change in slope of a given moment may increase at small scales either as a results of seeing a set with a higher dimension in projection or following the oscillation of a homogeneous fractal. Once the domain of the higher dimensional set is entered, it is hard to leave (without invoking a conveniently located inner cutoff). If the slope is observed to decrease toward smaller scales, it seems more plausible to attribute the change to a homogeneous oscillation.

Part II: Cosmic Lacunae

Self-similar structures have a long history in astronomy; the large scale distribution of matter in the Universe inspired one of the first self-similar models of nature. At the turn of this century, Fournier (1907) proposed a strictly self-similar mass distribution as a solution to Olber's paradox. The model describes a hierarchial universe, ordered at all scales though homogeneous at none. The dynamics and evolution of such a distribution have also been

discussed by Hoyle (1953). Our interest in the distribution originated in the hope of observing the oscillatory component of the scaling by examining the angular position of galaxies on the sky. The existence of such an oscillation was proposed by de Vaucouleurs (1970, 1971), who conjectured that the structure of the cosmos results in a clustering hierarchy of visible matter such that the number of clusters or aggregates of a given mass follows a scaling law, complete with a superimposed oscillation periodic in the logarithm of the length scale. While more modern distributions have been advanced (see Mandelbrot, 1977), the two primary features are demonstrated nicely in Fournier's example. With this simple model, the observational consequences of fractal distributions are explored and the effects of projection on the d_q spectrum are noted. Next, the arguments of Kida are applied to model the dynamics of the cosmic cascade. Such processes typically give rise to distributions with self-similar geometry. A method for constructing such distributions is introduced to show that a complex array of filaments and voids appear in these simple cascade models. The chapter is concluded with a report of the correlation integral of nearby galaxies as a function of angular separation. No oscillation is found.

Section 4.4: The Fournier Universe

The Fournier universe consists of a structured hierarchy, each generation of which consists of seven "stellar aggregates" located at the vertices and center of a regular octahedron (see Figure 4.2).

The model, a Cantor set embedded in three dimensions, is a homogenous fractal of topological dimension zero; with a similarity ratio equal to $1/7$ and a filling factor of 7 , its capacity dimension is equal to one.

Consider an n^{th} generation aggregate (n -aggregate) of scale ℓ_n . If $N(\ell_n)$ represents the number of n -aggregates observed, then

$$\begin{aligned} N(7\ell_n) &= 1/7 N(\ell_n) \\ N(\ell) &= \chi(\log \ell) \ell^{d_0} \quad \text{with } d_0 = 1 \end{aligned}$$

and we have the gist of de Vaucouleurs' intuition with this simple model.

The dynamics of these hierarchial models is described as a fragmentation cascade from an initially uniform density distribution. In the present case we consider a co-moving domain of the Hubble flow, specifically a box, B , whose sides define an outer scale ℓ_0 . We shall assume ℓ_0 sufficiently small that a Newtonian theory is applicable. During the first step of cascade, the initially uniform density of B clumps into aggregates of size ℓ_1 ; the cascade continues, the n^{th} generation consisting of N_n n -aggregates of size ℓ_n . For ℓ_{n+1} sufficiently less than ℓ_n , the impression of each generation remains observable. The population dynamics of such a process are described by the Kida equation

$$\frac{dN_n}{dt} = - \frac{N_n}{\tau_n} + \left(\frac{\ell_{n-1}}{\ell_n} \right)^\kappa \frac{N_{n-1}}{\tau_{n-1}} \quad (4.28)$$

where there are N_n n -aggregates with a lifetime τ_n and each n -aggregate produces $\left(\frac{\ell_n}{\ell_{n+1}} \right)^\kappa$ $(n+1)$ -aggregates.

We make the following assumptions:

- (1) The ratio of the characteristic lengths scale of n -aggregates to that of $(n+1)$ -aggregates is a constant, β , independent of n ; for contraction $\beta > 1$. That is

$$\ell_n = \beta \ell_{n+1} \quad \text{For all } n.$$

- (2) The characteristic time for each n -aggregate is much less than the Hubble time, and scales as a power of the relevant characteristic length. Assuming some outer time scale T we have

$$\tau_n = T \ell_n^\eta.$$

The additional assumption that the cascade has "run its course" before the current epoch implies

$$\frac{dN_n}{dt} = 0$$

so that

$$N_n = \left(\frac{\ell_{n-1}}{\ell_n} \right)^{\kappa-\eta} N_{n-1} \quad (4.29)$$

with solution

$$N_n = \ell_n^{-d} \chi \left(\frac{\log \ell_n}{\log \beta} \right)$$

where

$$d \equiv \kappa - \eta$$

and χ is periodic with period one. Suggested values of d range from one to three; while β has been argued on various "physical" grounds

to be equal to 5 by Hoyle (1953), equal to 7 by Fournier (1909), and equal to 10 by de Vaucouleurs (1971).

While we expect only the n -aggregates to be visible, one may include the exchange of mass with an invisible background. The assumption that this interaction is also scaling gives

$$M_n = \Phi M_{n-1} \quad (4.30)$$

where M_n is the totality of mass in n -aggregates and the interaction term Φ is

$$\Phi = \left(\frac{\ell_{n-1}}{\ell_n} \right)^\mu \quad (4.31)$$

In the absence of interactions with the background μ is zero; currently we cannot determine even the sign of μ .

If there is an interaction with a background mass, one would expect to detect it in the mass distribution among n -aggregates.

Since

$$M_n = \rho_n \ell_n^3 N_n$$

the density between different generations is related through

$$\rho_n \ell_n^{3-d} = \rho_{n-1} \ell_{n-1}^{3-d} \left(\frac{\ell_{n-1}}{\ell_n} \right)^\mu$$

where the rescaling of lengths by a factor of β eliminates any contribution from χ .

Substituting we find

$$\rho_n = \left(\frac{\ell_{n-1}}{\ell_n} \right)^{3-d+\mu} \rho_{n-1}$$

or

$$\rho \sim (\ell)^{-(3-d+\mu)}$$

With the additional assumption that the background mass is distributed uniformly, we have

$$M_n \sim \ell^{d-\mu} N_n = \ell_n^{-\mu} \chi(\log \ell). \quad (4.32)$$

The participation of the hidden mass is revealed in the mass distribution of the n -aggregates provided the hidden mass interacts more or less as above.

Section 4.5: A Geometric Construction

The Kida equation describes the dynamics of the cascade process; it suggests the structure will be fractal. We now introduce a generator for the construction of particular geometries. The procedure is straightforward and includes several distributions considered by other investigators as special cases. Despite the simplicity of the construction, the results have rich probability distributions. The output of this process could serve as the basis for Universe simulations similar to those of Soneira and Peebles (1978) who used nested, random distributions.

In the present case, begin by choosing a set of N 3-vectors g_i and a 3×3 generating matrix $G_{ij}(\sigma, \theta, \varphi)$ of the form

$$G = \sigma \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta)\cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \quad (4.33)$$

The zeroth generation consists of the $2N$ points 0s_j , located by

$${}^0s_j = \begin{cases} +g_j & j = 2i \\ -g_j & j = 2i + 1 \end{cases} \quad i = 1, 2, \dots, N \quad (4.34)$$

Defining the vectors $g_i^1 = G g_i$, we get the first generation

$${}^1s_j = \pm g_i \pm g_k^1 \quad \text{all combinations of } \pm. \quad (4.35)$$

Repeat the procedure with $g_i^2 = G g_i^1$ to form 2s . And so on. A set s is formed with retention if, at each generation, the set includes the union of all previous generations. Usually this will not be the case; only the new members of the n^{th} generation will be used in future iterations of the cascade as indicated in Equation 4.35 above.

Five iterations with the particular choice $\sigma = 1/7$, $\theta = 0$, $\varphi = 0$ with $N = 3$ and

$$g_i = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix} \right\}$$

produces the distribution shown in stereograph 1. Formed with retention, this is the Fournier Universe. By adjusting σ , the capacity dimension of the set may be altered to yield any value between zero and three. When the measure is distributed uniformly in each generation, all the D_q are equal and $f(\alpha)$ is constant.

Section 4.6: Projection Effects

A complete three-dimensional view of the local matter distribution in the Universe is currently unavailable to the observer. More accessible data consists of the angular position of galaxies as points on the celestial sphere. To simulate this in the Fournier case, position an observer at a 5th generation point 0. The observer at 0 records the distribution shown in equal area projection in Figure 4.3. A set mundane and intrinsically uniform as the Fournier universe yields many apparent structures in projection. Considering the correlation integral as a function of the angular separation ξ , the observed scaling has a large oscillatory component.

The effects of projection are more easily visualized by considering the members of the set which fall within the horizontal mid-plane of an octahedron (Figure 4.4). This set, called the equatorial section by Fournier, is also a homogeneous fractal; the filling factor is changed to five; thus

$$d_0 = d_q = \frac{\log 5}{\log 7} .$$

Since the similarity ratio is invariant under this procedure, the period of a ψ function of the set will not change. The two-dimensional equivalent to projection onto the sky is projection onto a circle, of consideration of only the angular position, ξ . Locally, in angle, the set will appear as if projected onto a line. Assuming the coordinate system shown in Figure 4.4, projection onto the x (or y) axis gives $\rho = 3$, $\sigma = 5$ while projection onto the line $x = -y$ yields $\rho' = 3$, $\sigma' = \sqrt{2} \sigma$; a homogeneous fractal when viewed in this

projection, may have a non-trivial d_q spectrum.

For non-zero values of θ and φ , these structures are apparent in the unprojected distribution. In the case $\varphi = 0$, $N = 1$ with

$$g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

the set lies in the $z=0$ plane. With the identification

$$s = \sigma e^{i\theta}$$

this set corresponds to the objects studied by Barnsley. In his notation the set consists of all points

$$\pm 1 \pm s \pm s^2 \pm \dots$$

Several probability distributions for this situation are shown in Plate 1; specifically, the cases $\theta = 23$ degrees, $\sigma = 0.90, 0.93, 0.96$, and 1.00 times the $\sqrt{2}/2$. The point illustrated here is that very complex conglomerations of filaments and voids are expected in fractal distributions.

Section 4.7 : Observations

Recent observations reinforce the view that the visible matter in the Universe is distributed in a very nonuniform manner. In the slices in which unprojected three dimensional data are available (see Lapparent, 1986), large voids are observed. These are presumably cross-sections of huge, relatively empty bubbles. While spherical projection of a distribution in 3-d may disturb the d_q spectrum, the simple examples above suggest that an oscillatory component may remain observable. The presence of an oscillation in the cosmic

distribution of visible matter would provide yet another new universal number — the period of the oscillation. In order to search for such an oscillation, we have examined the Zwicky and Shane Wirtanen catalogs. Analysis of the Zwicky catalog consisted of the direct evaluation of the correlation to a statistical sample of 3333 galaxies supplied by Groth. We determine

$$\nu = -1.8 \quad \text{in the range } 0.05 < \xi < 1.0 .$$

No significant oscillatory component was found.

We also considered determining the d_q from the Shane Wirtanen survey (see Peebles, 1973). Here we have not yet successfully dealt with the problem of variations from one plate to the next (this structure is apparent in color projections of the relative density) and the difficulty of taking into account the overlap of the plates.

Figure	Captions
4.1	The β model picture. This figure illustrates nested regions of activity. Each active region of size l is composed of three smaller, more intensely active regions.
4.2	A three generation (7^3 point) approximation of the Fournier universe. In this model, matter is nested in a strictly self-similar hierarchy.
4.3	An equal area projection of the data in Figure 4.2 as viewed from the point $(\sigma + \sigma^4, \sigma - \sigma^4, \sqrt{2}(-1 - \sigma^2 - \sigma^5))$ looking in the direction of increasing z (upwards).
4.4	The "equatorial section" of the Fournier Universe, this section corresponds to the $z=0$ plane.

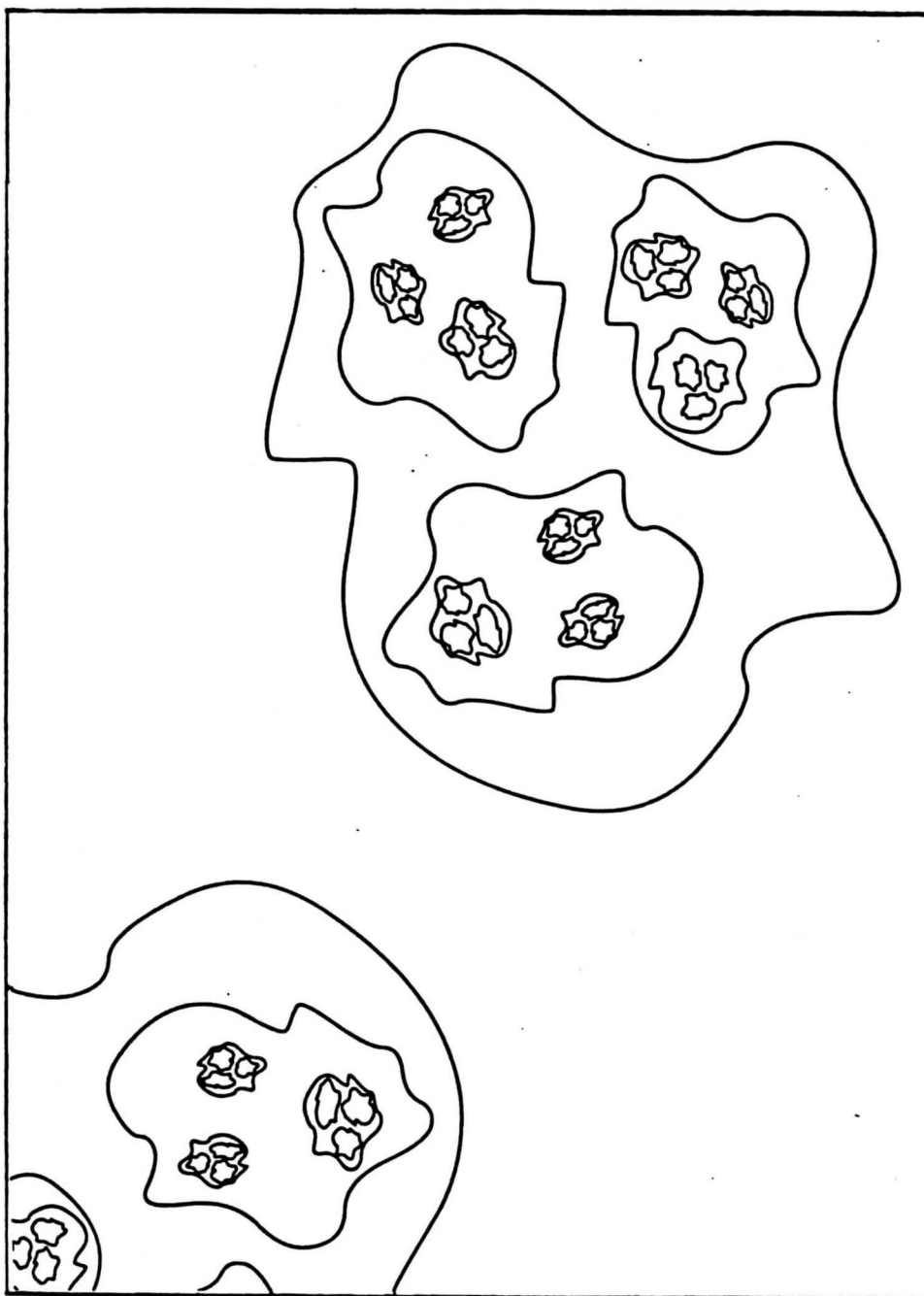


Figure 4.1

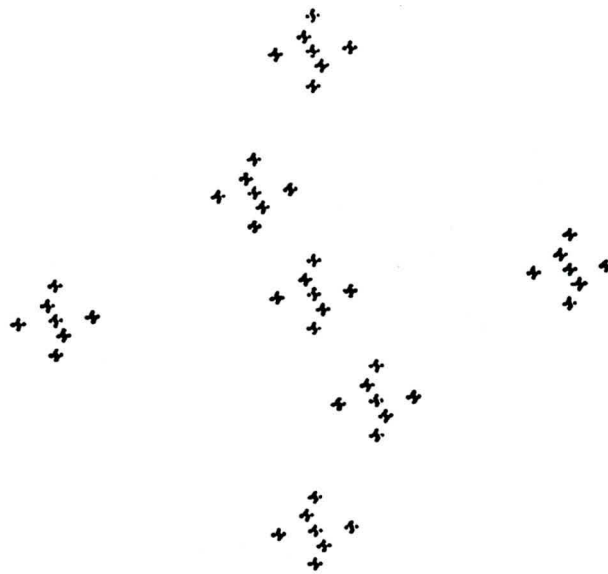


Figure 4.2

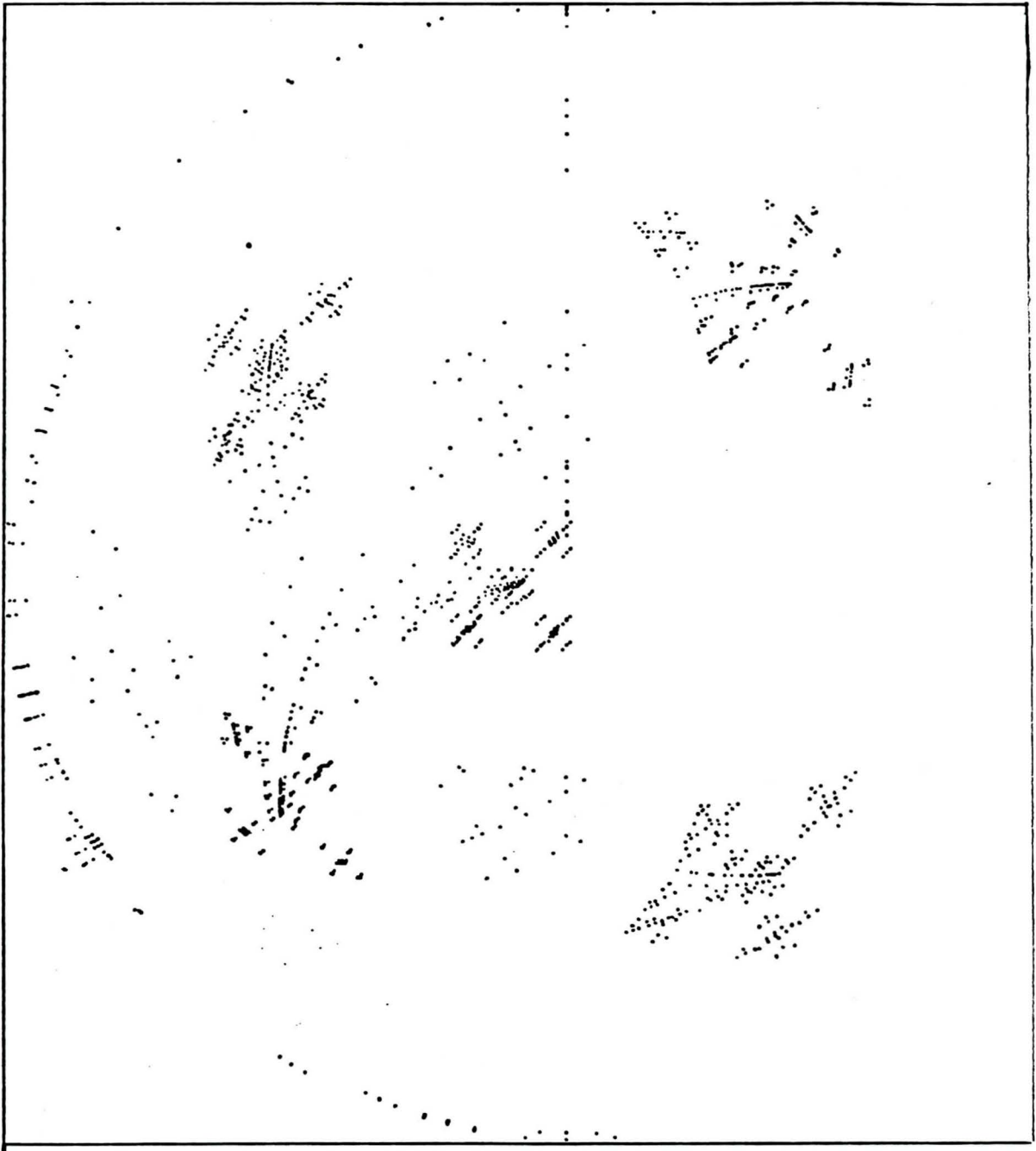


Figure 4.3

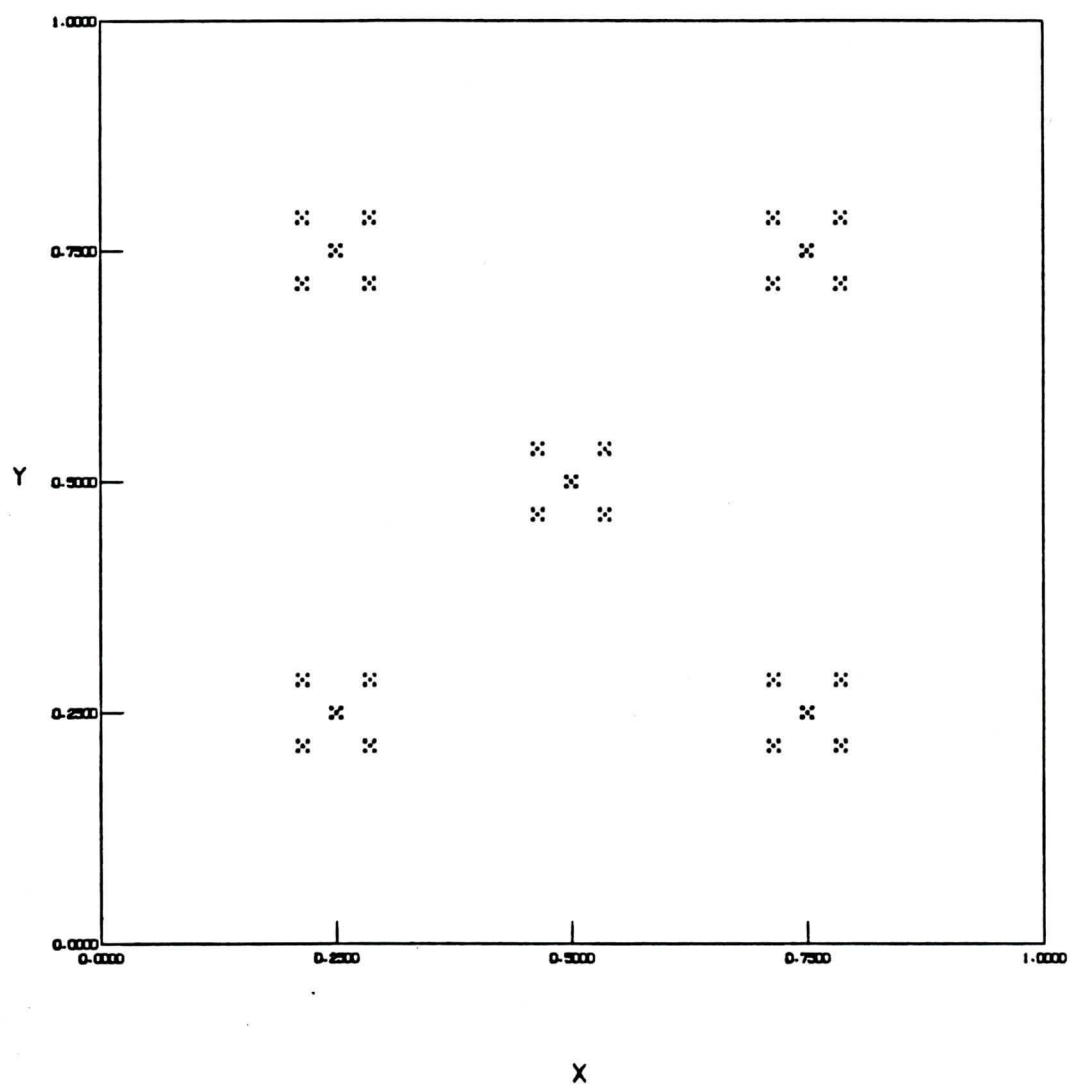


Figure 4.4

Chapter 5 PARTICULATE DISPERSAL IN A TIME-DEPENDENT FLOW

Introduction

In this final chapter we develop a model for the motion of tracer particles suspended in a time dependent flow. The model demonstrates the stability of the suspension of negatively buoyant particles to time dependent perturbations and the exponential growth in the length of a tracer line in a nonturbulent flow field. Treated as a dynamical system, the model equations differ from those discussed in chapters 2 and 3 due to their area conserving property. Examination of sedimentation in this model reveals highly structured probability distribution which we believe are a generic feature of area preserving maps and flows. A discussion of the accumulation of probability density with several area preserving maps is given in Appendix 4 while a mapping for the model discussed in this chapter is developed in Appendix 3.

The behavior of particles suspended in a moving fluid is of interest in many areas of geophysics; in some problems, a better understanding of this behavior is a prerequisite for further progress. Three examples in which this phenomena is relevant are: (1) precipitation formation and the lifetime of volcanic aerosols in the atmosphere (Rogers, 1976), (2) suspension of plankton in the sea (Stommel, 1949), and (3) the evolution of growing crystals in a convecting magma chamber (Huppert (1984), Maxey and Corrsin (1986)). In each of these realizations, negatively buoyant particles may remain in suspension for a time far exceeding naive dimensional calculations and it is this aspect of the problem which we intend to

study here.

As an initial step toward understanding such phenomena, we examine the motion of particles in a simple laminar flow which is periodic in time. Stommel (1949) developed the theory for the case of steady rolls while experimental investigations of a similar steady flow field have been performed by Tooby *et al.* (1977). Recently, experiments by Gollub (1987) have probed particle behavior in a time-dependent convective flow similar to the flow considered here. The correspondence between a time-dependent surface flow and an area preserving mapping has been developed by Chaiken *et al.* (1986, 1987). In Chaiken *et al.*'s experiments, the correspondence between the laboratory flow and the mapping is sufficiently accurate that the mapping may be used to predict features of the flow.

The motions of fluid particles in time-dependent flows have been studied by Aref (1984), who has modeled the stirring of a tank of fluid by alternately flashing, point vortices. The novelty of our approach is to allow a simple two-dimensional incompressible flow to become periodic in a smooth manner (Smith, 1984). Since the velocity field is completely determined as a function of position and time, the phase space is effectively three-dimensional, the lowest dimension permitting chaotic behavior. Such behavior is in fact observed. In addition to regions of retention and simple fallout, regions of chaotic particle motion are observed. The presence of chaotic motion holds deep implications for the sedimentation properties of the system. In the physical surface flow studied by Chaiken *et al.* (1986,1987), chaotic motion is observable through the

rapid evolution of tracer distributions. These effects are relevant in many fluid systems.

Section 5.1 Steady Background Flow

Consider small bodies immersed in a fluid whose motion is described by a stream function $\Psi(x,y,t)$. The particle motion is governed by Newton's second law which takes the form

$$\begin{aligned} m \dot{x} &= -6\pi a\mu(\dot{x} - \Psi_y) \\ m \dot{y} &= -6\pi a\mu(\dot{y} + \Psi_x) - m'g \end{aligned} \quad (5.1)$$

where a is the particle radius, μ the viscosity of the fluid, g the acceleration of gravity, m the mass of the particle and the effective mass, m' adjusted for buoyancy effects, is given by

$$m' = \frac{4}{3} \pi a^3 (\rho - \rho_f),$$

where ρ is the particle density and ρ_f the density of the fluid. The coordinate system is defined with the y axis vertical.

If particle inertia is negligible, the particle trajectories also follow a stream function. That is, the trajectories are solutions of

$$\begin{aligned} \dot{x} &= \varphi_y \\ \dot{y} &= -\varphi_x \end{aligned} \quad (5.2)$$

where the particle stream function is

$$\varphi(x,y,t) = v_s x + \Psi(x,y,t) \quad (5.3)$$

and v_s , the Stokes velocity, is given by

$$v_s = \frac{m' g}{6\pi a\mu} = \frac{2a^2(\rho - \rho_f)g}{9\mu} . \quad (5.4)$$

The Stokes velocity is the terminal velocity obtained by a particle in free fall through a quiescent viscous fluid.

The reduced equations 5.2 define a Hamiltonian dynamical system with one degree of freedom. The particle stream function φ is the Hamiltonian; its x and y location form a conjugate coordinate momenta pair. When φ is independent of time, topological constraints prevent trajectories in such a system from displaying chaos. A time-dependent stream function corresponds to a nonautonomous Hamiltonian system; this case resembles a system with a three-dimensional phase space. Such a system admits qualitatively different (particle) trajectories.

The effect of steady convective rolls on the motion of a small, slowly sinking body was first investigated by Stommel (1949), who was intrigued by the observation that the yield of plankton tows taken along the direction of the wind was much more variable than that of tows taken perpendicular to the wind. Long cellular rolls generated by wind stresses (Langmuir, 1938) are often observed in oceans and lakes where they are made visible by floating debris trapped above subduction zones. Treating the plankton as negatively buoyant point masses, Stommel considered a vertical cross-section taken parallel to the direction of the wind. In this cross-section, the flow is approximately two-dimensional; treating it as such, Stommel chose the stream function

$$\Psi(x,y) = A \sin x \sin y . \quad (5.5)$$

Streamlines of this flow are shown in Figure 5.1. A similar flow field is associated with the first unstable mode of Rayleigh-Bernard convection (Chandrasekar, 1961). Substitution of Equation (5.5) into (5.3) yields Stommel's result: a stream function for the particle motion of

$$\varphi(x,y) = A \sin x \sin y + v_s x \quad . \quad (5.6)$$

In order to examine how the system behaves for a variety of parameter values, we normalize the Stokes velocity by the maximum fluid velocity and obtain

$$\gamma = \frac{v_s}{A} \quad . \quad (5.7)$$

Particles for which $\gamma = 0$ are neutrally buoyant and they follow the streamlines of the fluid. All particles for which $\gamma > 1$ fall through the cells with horizontal displacement (in the x direction) as shown in Figure 5.2. The sinking particles have minimum vertical velocity in regions of maximum upward fluid velocity, this increases their residence time in the area, and hence their horizontal displacement due to the horizontal component of the velocity of the background flow. A similar type of behavior occurs for particles with $\gamma < -1$, which rise through the cells. For an intermediate value, $-1 \leq \gamma \leq 1$, there exists a region of retention in which particles will execute closed orbits, remaining suspended in the cell. The boundary of this region is delineated by the largest closed orbit within the cell; here, the particle stream function has the same value it takes along the cell border. In the steady case, this orbit is

infinitesimally close to the upward flow at the cell boundary. Motion along the cell boundary corresponds to separatrix motion in a pendulum.

Stommel concluded that this simple picture explained why the plankton tows in the direction of wind were less variable than those taken across the wind. The large variations in yield observed in tows taken parallel to the rolls distinguish those that sample a region of retention from those that are taken in relatively plankton-free regions between them. Tows perpendicular to the rolls sample both the plankton-dense and plankton-free regions and hence are more uniform.

Without loss of generality, we consider only negatively buoyant particles which are initially within the cell with a fluid stagnation point at $x = 3\pi/2$, $y = 3\pi/2$ (the upper right quadrant of Figure 5.1). Denote the stagnation point of the particle motion by x_s . At this point the fluid velocity is upward in direction and equal in magnitude to the settling velocity of the particle. Therefore

$$x_s = \sin(\gamma) \quad (5.8)$$

For a given value of γ , the orbit of a retained particle is uniquely identified by the location of its right most crossing of the line on which the horizontal (x) component of the fluid velocity vanishes (that is, the line $y = 3\pi/2$). The value of x at this point is denoted x_r . In steady flows, paths of particles with different initial positions do not cross unless they follow the same streamline (the phase space is two-dimensional).

Trajectories are determined by integrating equations (5.6)

numerically with a sixth order Runge-Kutta scheme using subroutine DVERK from the International Mathematics and Statistics Library (IMSL, 1978). Integration of trajectories in the steady flow is straightforward. The time steps used were typically on the order of $\Delta t = 0.01$. Trajectories of integrable orbits were not sensitive to the precise value of Δt near this choice.

First consider particles located within a region of retention. Figure 5.3 shows the period of the particle orbit as a function of x_r for several values of γ . Near the fluid stagnation point ($x = 3\pi/2$, $y = 3\pi/2$), fluid parcels (and $\gamma = 0$ particles) are in solid body rotation with period $P = 2\pi$. The orbital period increases with increasing x_r , becoming infinite for a particle following the orbit along the ascending cell boundary. For a given value of γ , particles near the particle stagnation point have the minimum x_r and lowest period orbits. As γ increases, the period of these tightest orbits ($x_r \approx x_s$) also increases. All particles with $x_r > x_s$ are retained.

The increase of the minimum period with γ may be understood as follows. On translation of the origin to the fluid stagnation point, the particle stream function becomes

$$\varphi(x,y) = \cos x \cos y + \gamma x \quad . \quad (5.9)$$

Expanding the velocity in a Taylor series about the particle stagnation point $\mathbf{x}_s$, we have, to leading order,

$$\dot{\mathbf{x}}(x,y) = \dot{\mathbf{x}}(x_s, y_s) + (x - x_s) \left. \frac{d\dot{\mathbf{x}}}{dx} \right|_{\mathbf{x}_s} + (y - y_s) \left. \frac{d\dot{\mathbf{x}}}{dy} \right|_{\mathbf{x}_s}$$

$$\dot{y}(x,y) = \dot{y}(x_s, y_s) + (x - x_s) \left. \frac{d\dot{y}}{dx} \right|_{x_s} + (y - y_s) \left. \frac{d\dot{y}}{dy} \right|_{x_s}$$

or

$$\begin{aligned}\dot{\tilde{x}} &= -\kappa \tilde{y} \\ \dot{\tilde{y}} &= -\kappa \tilde{x}\end{aligned}\tag{5.10}$$

where

$$\begin{aligned}\tilde{x} &= x - x_s \\ \tilde{y} &= y - y_s \\ \kappa &= \cos(x_s)\end{aligned}\tag{5.11}$$

and $\mathbf{x}_s \equiv (x_s, y_s) = (\sin^{-1}(\gamma), 0)$.

Particles near the particle stagnation point are in solid body rotation with period

$$P = \frac{2\pi}{\kappa}\tag{5.12}$$

This is the line in Figure 5.3. The cross marks along this line represent the locations of the stagnation points for numerical calculations with the parameter values specified in the caption.

In discussing sedimentation, it is useful to distinguish the particles which remain in the original cell from those which do not. A particle which passes through the bottom cell boundary is said to "fall out" of the cell, as opposed to a "retained" particle which does not cross a horizontal cell boundary. The horizontal motion of the particles is strictly that of the fluid; contours of zero horizontal fluid velocity (the lines $y = n\pi$, $n = 0, 1, 2, \dots$) are barriers which the particles cannot cross.

The symmetry of the flow field requires that the net effect of the fluid flow on the settling velocities, averaged over all space, is

zero. It is more interesting to examine the effect in the region of retention or the region of fallout separately. The area of the region of retention may be calculated as follows. Consider the region $0 \leq x \leq \pi$, $\pi \leq y \leq 2\pi$ (see Figure 5.2 a). Compute the area of the region of retention by integrating the contribution at each value of y from the point at which the streamline of value zero leaves the y axis until it returns. By symmetry this area is divided evenly by the line $y = 3\pi/2$. Thus

$$R_\gamma = \frac{2}{\pi^2} \int_{y_{\min}}^{\frac{3\pi}{2}} dy \, x_{\max}(y) \quad (5.13)$$

where $x_{\max}$ is defined by

$$\psi(x_{\max}, y) = 0$$

or

$$\gamma x_{\max} = -\sin(x_{\max}) \sin(y). \quad (5.14)$$

The fraction of the cell occupied by the region of retention is shown as a function of γ is shown in Figure 5.4. As expected, the fraction of area in which the particles are retained decreases from one for neutrally buoyant particles to zero for particles whose Stokes velocity is equal to the maximum fluid velocity. The region contains half the cell area for

$$\gamma \approx .35 .$$

Section 5.2: Phase Wrapping

Before considering a time-dependent streamfunction, we examine the behavior of a cluster of particles introduced into the steady

flow of the type examined above. As a line of particles initially on the upper cell boundary (completely outside the region of retention) falls through a series of cells its length increases linearly with time. A line which is initially located entirely within the region is rolled up by the flow as shown in Figure 5.5. The density of particles, when projected onto a horizontal plane, will evolve as in the upper panels of Figure 5.6. Visible particles distributed in a fluid roll, when viewed from above, may take on the appearance of the lower panels of this figure. The well defined structures in the projected density mimic the familiar phenomenon of vortex roll-up in suitably dyed fluids.

A similar behavior has been discussed by Quin (1984) in a study of stellar dynamics. Quin follows the motion of stars from a light spiral disk as they interact with the gravitational potential of a massive elliptical galaxy. While shells are apparent in his simulations when the data are plotted in physical space (and in the observations), they are more striking when the data is viewed in a velocity-position space. For this reason Quin calls the phenomena phase wrapping.

Stommel motivated his original study with a discussion of patterns formed in the sea. As shown by the studies of Quin, the problem has implications over a large range of scales; it may bear on such questions as dune formation in shallow water beaches, though the inertial effects may be significant in such a case.

Section 5.3: Experimental Observations

The motion of small negatively buoyant spheres in an axisymmetric flow is easily observed in the laboratory.¹ An apparatus consisting of a cylindrical tank filled with a high viscosity fluid and mounted with its axis horizontal is sufficient. As the cylinder is rotated about its axis, the fluid quickly attains solid body rotation. Small spheres placed in the ascending flow are observed to follow nearly circular orbits about the particle stagnation point. If small air bubbles (which are positively buoyant) are present, they are observed to execute similar motion in the descending fluid on the opposite side of the tank. A detailed investigation of this type of system has been performed by Tooby, Wick, and Isaacs (1977), using a variety of test particles and rotation speeds. On time scales long relative to the fluid motions, the orbits evolve, the density and diameter of the test particle determining the specific manner in which its orbit evolves. The instability of closed orbits is due to inertial effects and the influence of the walls on finite diameter test particles. Particle-particle interactions are also observed to produce large perturbations in particle motions (J. Whitehead, private communication). In the cases of precipitation formation and magma crystal growth, the properties of a single particle change with time influencing the particle's motion, which in turn feeds back upon the particle's growth and the surrounding fluid. None of these

¹ The following observations were made with J. Whitehead and B. Frazel at the Woods Hole Oceanographic Institute (see Smith, 1984).

complicating effects were considered in section 5.2 nor will be in what follows.

Experimental techniques of flow visualization (e.g. Corrsin, 1950) rely on tracers assumed to be passive. The rate of growth of material lines in a turbulent flow is expected to be exponential and lead to the development of fractal structures (Sreenivasan and Meneveau, 1986). For nonideal tracers, these behaviors occur in an unsteady laminar flow (Smith and Spiegel, 1986), leading to the formation of self-similar distributions.

Section 5.4: Time Dependent Flows

To consider flows smoothly varying in time, we let

$$A(t) = 1 + \epsilon \sin(\omega t) \quad (5.15)$$

With this choice, the phase space of system 5.5 becomes three-dimensional and the possibility for chaotic behavior arises. One could consider more spatially complicated flows by superimposing additional periodic structure periodic with half the wavelength of this flow. Repeating the process with additional, smaller-length scales would yield a flow similar to the β model. As shown below, the particle patterns produced by the single length scale are already quite intricate.

The presence of chaotic motion is immediately felt in attempts at analytic solution through perturbative methods where the problem of small denominators destroys the convergence of the solution for many initial conditions (see Lichtenberg and Lieberman (1983)). Here, we

continue to study the system via numerical simulation.

In most instances, the trajectories in three dimensions are either projected onto a plane or a two-dimensional surface of a section is taken (see Chapter 1). The situation is simplified in our case because the flow is strictly periodic in time; stroboscopy at the frequency of the fluid motion is equivalent to taking a surface of section. Runge-Kutta integration schemes are ideally suited for this method. Even so, numerical integration is a resource consuming process. A two dimensional area preserving map able to simulate long evolutions efficiently was constructed for this system. This mapping is described in Appendix 3.

We consider the effect of small ϵ on the motion of a particle well within the region of retention for the corresponding $\epsilon = 0$ flow. For ϵ equal to zero, the particle moves on a roughly circular path. For small ϵ , the particle will, initially, oscillate about its steady flow orbit with frequency ω (see Figure 5.8a). The asymptotic stability of this motion for non-zero ϵ was not established until 1963 (see Moser, 1968); the nonlinearities involved make the question of the stability of this system similar to that of the three body problem.

For small ϵ , particles well within the region of retention display three distinct classes of motion. The first corresponds to oscillations about the steady flow path. When a trajectory of the type (e.g.: Figure 5.7a) is viewed in section, the orbit again fills out a closed curve as shown in Figure 5.7b. By strobing at different phases it is seen that the particle winds about on a torus in phase

space. A single point in phase space defines the future evolution of the system; thus when a torus is covered densely, all initial conditions within a torus remain within it. In section, the tori appear as closed curves; initial conditions within closed curves on a Poincaré section are trapped there. This intuitive result holds for two-dimensional tori in three-dimensional space and therefore our system. Note, however, that by the definition of topological dimension (see Poincaré's quote in §1.1) this argument fails in spaces of dimension greater than three, giving rise to a slow spreading of trajectories known as Arnold diffusion (Lichtenberg and Lieberman, 1983).

Viewed stroboscopically, the orbits of nonzero γ particles near their particle stagnation point will move through an angle $\Delta\theta_i$ between the $(i-1)^{\text{th}}$ and i^{th} recording and will, for $\Delta\theta$ sufficiently irrational, fill out a closed curve². Initial conditions for which $\Delta\theta$ is a rational angle resonate with the strobing frequency so that a finite number of points are observed in a repeating sequence. For nonzero ϵ , not too large, the majority of particles of the first type continue to move on trajectories that are tori. Two different types of motion are observed for particles whose unperturbed period is near resonance with that of the fluid: one corresponds to motion on islands, the other to chaos. For the former, in each revolution about x_s , the particle follows one of several distinct pathways. Which path is executed depends on the phase of the flow as the

²If $\Delta\theta$ is not sufficiently irrational, the trajectory may describe a Cantor set.

particle crosses the x axis from below ($x > x_s$). This type of motion is shown in projection in Figure 5.8a. The clear regions shown remain clear, with the trajectory being restricted to, and slowly filling, the outlined region. In this example two revolutions about the stagnation point, one along each branch, occur in three cycles of the background fluid flow; the particle executes three revolutions about x_s until its crossing of the x axis occurs at approximately the same position and phase with respect to the fluid flow period. When such a trajectory is strobed at the fluid flow frequency, islands are observed (Figure 5.8b). Particle motion on these islands is stable. The particle visits every island in turn, slowly delineating each. Poincaré sections taken at different phases are shown in Figure 5.10. The islands slowly deform and rotate in the direction of particle motion until, one full fluid period later, particles initially on island 1 (2,3) have taken positions on island 3 (1,2). Recalling that these figures are cross-sections of a three-dimensional phase space, it is seen that this motion takes place on a torus which is stretched and twisted, closing on itself in three fluid periods. Particle paths wind around on this torus. Plotting t as z results in the three-dimensional reconstruction of Figure 5.9a, while the periodic boundary conditions are more naturally reflected in Figure 5.9b where the trajectory is shown in polar coordinates (r, θ, z) related to (x, y, t) by the transformation

$$\begin{aligned} r &= x + r_0 \\ \theta &= \omega t \\ z &= y \end{aligned} \tag{5.16}$$

where the constant displacement r_0 is chosen relative to the point at which the trajectory crosses the line $y = 3\pi/2$ at zero phase of the fluid. Initial conditions sufficiently inside or outside the island ring (relative to the particle stagnation point) wind about a single torus. Particles initially located between the islands at the same distance from the stagnation point are observed to display chaotic motion. For sufficiently small ϵ , these trajectories remain in the vicinity of the island ring.

The general structure of this pattern is understood (see Lichtenberg and Lieberman, 1983). The particle stagnation point corresponds to a stable elliptic point. For nonzero perturbations, trajectories about the stagnation point which resonate from a series of elliptic and hyperbolic points. The (stable) elliptic points correspond to the centers of islands, the (unstable) hyperbolic points to chaotic regions. Surrounding each of these equilibrium points is another generation of elliptic and hyperbolic points. The cascade continues to all scales. For the simplest integrable orbit, each passage through the plane corresponds to a rotation about the particle stagnation point, θ grows in a regular manner, filling the interval $-\pi < \theta \leq \pi$ densely and variations in radius are small; as a function of θ , the radius is single-valued. In the island case, there are bands of values at which θ is never observed, r is a double-valued function of θ . In a chaotic trajectory on which the particle falls out of the cell. This results in values of r greater than $\sqrt{2}\pi$ and biases θ toward large angles and $\Delta\theta$ toward small values. For all chaotic trajectories, r is multi-valued function of θ .

The appearance of this structure corresponds to a qualitative change in the region of retention when the flow becomes temporally periodic. In the case of steady flow, particles arbitrarily near an up-flowing branch are retained, completing closed orbits. In a periodic flow, the region of retention detaches from the cell boundary. Particles initially in the band of separation will often remain in the cell for many revolutions, but (most) eventually fall out. For small ϵ , a particle oscillates about its equilibrium path. As long as these excursions are completely within the region of retention, the particle will remain in the cell. A particle which oscillates to a point outside the retention region may remain in the cell for a time, depending on where in the cell it is when it crosses the boundary. If the frequencies of these two oscillations are incommensurate, these particles will eventually cross the boundary near the bottom of their trajectories and fall out of the cell. For some initial positions, a resonance between the fluid oscillations and the particle motions occurs which tends to stabilize the particle, producing stable islands in the chaotic sea. Gaps corresponding to the location of this type of island chain are visible in several of the figures (eg. Figure 15). Figure 5.11 is a histogram of the frequency of various residence times, it is clear that particles take much longer to fall through a series of rolls than would be predicted from the steady case.

The strobed paths for a variety of initial positions are shown in Figure 5.12. Counting from the left, the first particle (1.1π , 1.5π) falls through the cell, oscillating about the strobed path shown.

This line presents a barrier which other particles do not cross; hence the open region in the lower left-hand side of the figure. (The motion of initial conditions in this open area is symmetrical with that of the trajectories shown.) As the initial position of the particle is moved to the right along $y = 3\pi/2$, the trajectory breaks up into islands, which in turn become part of the chaotic sea shown in the figure. Particles initially in this area are often trapped in a cell for many revolutions and then fall (drifting) through the next cell (and often several more) before becoming re-entrained. Embedded in this chaotic sea are regions avoided by the falling particle. The largest of such areas is the region of retention containing the particle stagnation point. Islands are observed in this and several other of the barren regions where the particle motion is such that they are stabilized against sedimentation as discussed above.

It is this stabilization which complicates the calculation of an average fallout velocity. In the vicinity of every island structure there exists a higher-order island chain, each member of which is surrounded by yet another chain. A particle in the chaotic sea which moves into the region will remain outside these islands but may remain in this "reef" for an extended period of time. When the probability distribution of a Poincaré section is viewed, the reef areas that have been visited stand out as regions of high probability. The long runs required to get reasonable statistics are more easily obtained with conservative maps. The structure of these strange accumulators is described in Smith and Spiegel (1987), which is reproduced here as Appendix 4.

The effect of increasing ϵ with fixed v_s is shown in Figures 5.13. Here, the area of the chaotic region is observed to grow with ϵ . Individual islands are observed to break up, allowing particles to pass within, but also forming the intricate reef regions of the accumulator. The evolution of the upper island of Figure 5.13b is shown in detail as ϵ increases in Figure 5.14.

For moderate values of δ , the chaotic region B occupies a large area of the section. The chaotic sea has holes on all scales corresponding to island chains. Sets like the shown in Figure 5.15 have been called fat fractals by Umberger and Farmer (1985), who proposed a scaling relation which can be used to estimate the total area visited by a chaotic trajectory.

Before considering the motion of swarms of particles, we note that particle motion at large ϵ is qualitatively different from that of small ϵ . Quasi-stable regions in a flow with large ϵ (≈ 10.0) have been found where the particle revolution period is slightly less than half the fluid period. Such a particle, initially near the top of its orbit will be swept around by the strong flow, again to near the top of the cell, as the flow weakens it will sink down toward the cell center and then be swept around in the opposite direction sense by the second half cycle of the fluid flow. Often the radius of the particle from the stagnation point will increase until the particle falls out, however some initial conditions particles have been observed to be carried up above their initial points and then dropped back near their original position (and phase). In this manner, the particle is retained in the cell for the entire observation (= 300

fluid periods).

Figure 5.16 shows, at several different times, the locations of 513 $\gamma = 0.25$ particles that were initially spread along the line $y=2.0$ in the x interval $(1.85, 1.90)$ as they evolved in a flow with $P = 4.5$ and $\epsilon = 0.50$. Particles from this interval are outside any region of retention; they spread in vertical extent, but stay within the original cell width. In the final frame, the particles are spread over 9 cells. In this case, the abscissa is the value of y (not mod 2). We wish to see whether a fractal structure develops. However, there are not enough points in Figure 5.16 to do this. In Figure 5.17, we show the results of a calculation designed to suggest the detail in the loops of Figure 5.16. In this calculation, the periodic boundary conditions in y are applied, showing only the upper half of a cell. There are long intertwined filaments extending into the lower half of the cell where the segment was stretched exponentially. This poses a resolution problem.

The complexity of this distribution is quantified by examining the vertical final separation of particles which were initially nearest neighbors. In Figure 5.18, we show the (scaled) y -coordinate of each particle as a function of x_0 , the initial x -coordinate, for a sequence of times. Portions of these curves with very large slopes correspond to initial line segments of particles that have been stretched over many cell heights. The stretching and folding in x - y space of the original line of particles produces the self-similar structure seen in Figure 5.15. Braking the x_0 coordinate into steps of size Δx_0 and determining the length, L , of each curve as a

function of Δx_0 , we fit L to the formula

$$L \propto (\Delta x_0)^{-\sigma}. \quad (5.17)$$

As a function of time, σ increases rapidly to a plateau. The precise form of this evolution depends on how we perform the calculations; the plateau at $\sigma=0.76$ is characteristic and appears to be independent of the initial particle density. This plateau does not survive indefinitely at fixed resolution. As expected, it persists longer if the initial density is higher. We estimate the value $\sigma=0.76$. We do not have a good way to determine the precision of this result but, on the basis of many such calculations, we would estimate that the internal errors are less than 10%. σ is not very sensitive to ϵ and γ for ϵ in the range 0.1 to 0.8. For ϵ less than about 0.1, the effect of time dependence is so weak that the particle spreading is very slow, while for large ϵ the integration time steps required become prohibitively small. As the scaling structure of Figure 5.19 develops, the length of the particle line increases rapidly. A plot of this length with time shows exponential growth (see Figure 5.20), though the fluid flow is not turbulent; exponential stretching of a material line of tracer is not a sufficient condition for turbulent motion in the host fluid.

Section 5.5: Conclusions

We have constructed a model for the behavior of particles

suspended in a time-dependent flow. It is observed that negatively buoyant particles may remain suspended in the flow for a time exceeding the estimate based on a sedimentation at the Stokes velocity. More relevant from a dynamical systems point of view is the observed self-similar inhomogeneities which develop and persist in the probability distribution of a single trajectory. The self-similar structure of these inhomogeneities pose another source of difficulty in determining the correlation integral from an experimental time series. Specifically, the application of the Grassberger-Procaccia algorithm to a finite sample of an trajectory which is (asymptotically) space filling in two dimensions may indicate a value of the correlation exponent which is significantly less than two.

Finally, it is demonstrated that a material line of tracers may develop a complex, self-similar distribution in a smooth flow field. The development of this pattern requires the rapid stretching and wrapping of the tracer line. The exponential growth of a tracer material line is not a sufficient condition for turbulent flow.

FIGURE CAPTIONS

Figure	Caption
1	Fluid streamlines of the steady flow corresponding to Equations 5.5. A streamline corresponds to a contour of streamfunction; each streamline shown is labeled with this value.
2	Particle streamlines in the steady flow case for several Stokes velocities as shown. Each streamline is labeled with the local value of the particle streamfunction.
3	The period of retained particle orbits as a function of initial position for several Stokes velocities.
4	Area of the region of retention as a function of γ .
5	Phase wrapping on a line of particles within the region of retention for the flow parameters $v = 0.2$. The distributions correspond to times (a) $t = 0$, (b) $t = 18$, (c) $t = 36$, (d) $t = 54$.
6	The particle densities integrated vertically for the four states in the previous figure. The upper panels show the particle number explicitly while the lower panels are 'dust plots' simulating the appearance of suspended particles viewed from above.
7	A trajectory corresponding to motion on a torus (a) projected and (b) strobed.
8	A trajectory corresponding to motion on a period three island chain (a) projected and (b) strobed.
9	A three-dimensional reconstruction of the trajectory projected in Figure 8a. The reconstruction on the left shows the winding of the islands over one period of the background flow. On the right, time has been taken as an angular variable as described in the text. In this way the continuous nature of the trajectory is apparent.
10	Poincaré sections of the trajectory of Figure 8 taken at several phases. These section correspond

to planes of constant z in Figure 10 and show the islands at different phases.

- 11 A histogram of the residence times per cell for a particle in the chaotic fallout region. This trajectory was taken with $P = 4.5$, $v = .25$.
- 12 Strobed motions of six $\gamma=0.25$ particles in the time dependent Stommel flow, $\epsilon = 0.5$, $P = 4.5$
- 13 Poincaré section of the $\gamma=0.25$ paths with $P=4.5$ and increasing ϵ , (a) $\epsilon=0$, (b) $\epsilon=0.25$, (c) $\epsilon=7.5$.
- 14 The effect of increasing ϵ as shown in the "ear" region, $\gamma = 0.25$, $P = 4.5$.
- 15 An extended integration (3000 P) for seven initial conditions in the chaotic region; $\gamma=0.25$, $P = 4.5$.
- 16 Evolution of a line of particles initially on the upper boundary of a cell. In their initial configuration, the particles are labeled with letters alphabetically with increasing x coordinate. The position of the line is shown after (a) 2, (b) 5, (c) 6 (d) 7, and (3) 21 periods of the background flow.
- 17 A surface of section for the line of particles described in Figure 5.17. Only the upper half of the cell is shown.
- 18 Time development of a fractal showing the vertical displacement as a function of initial horizontal position at several times.
- 19 Final time sequence of Figure 19 at $T=24$ fluid periods.
- 20 Length of line segment as a function of time.

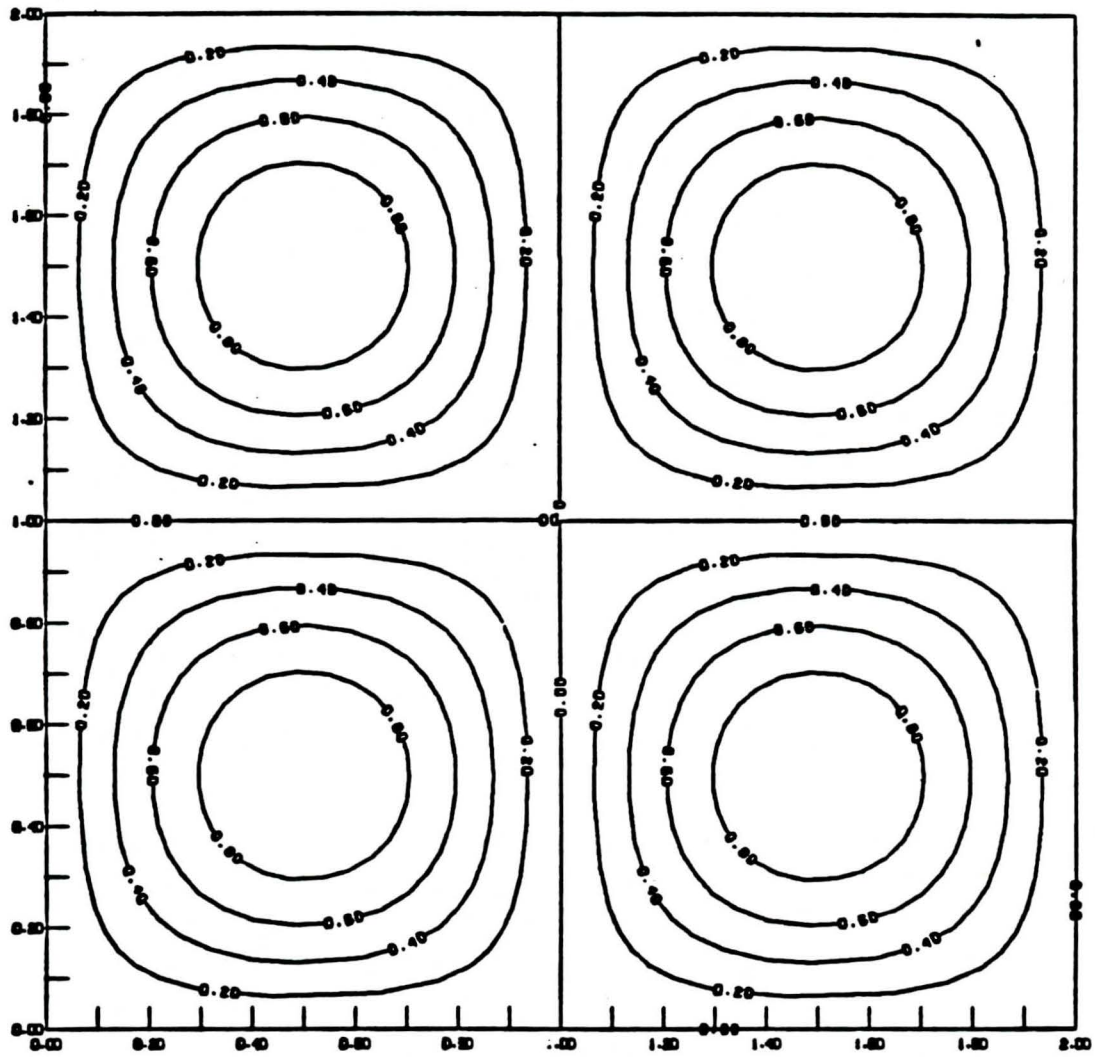


Figure 5.1

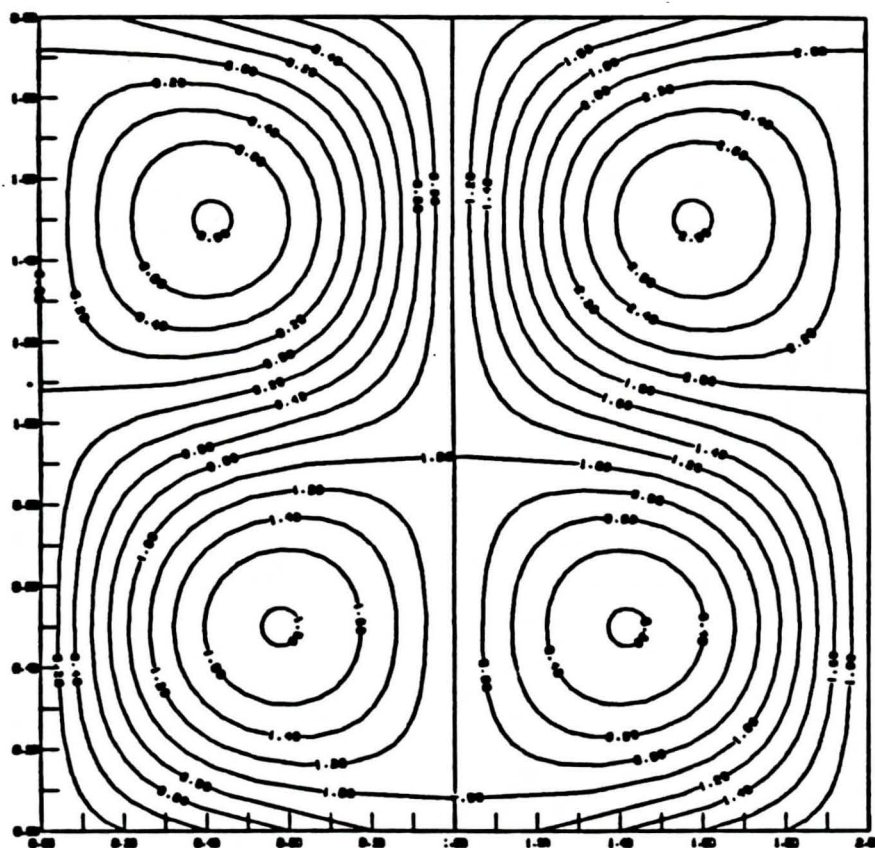
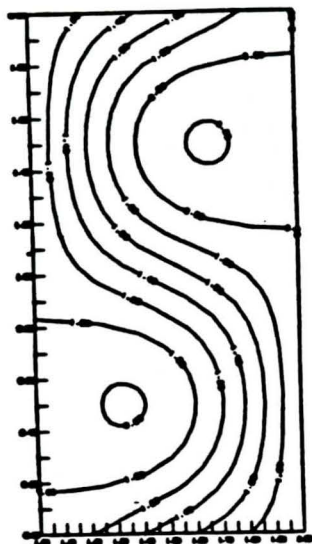
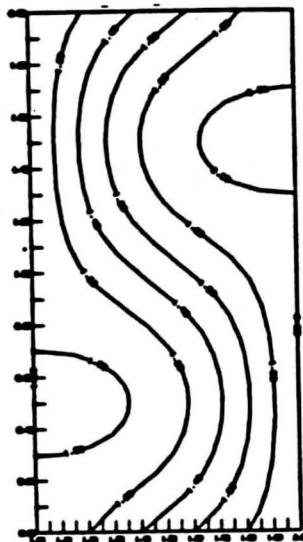
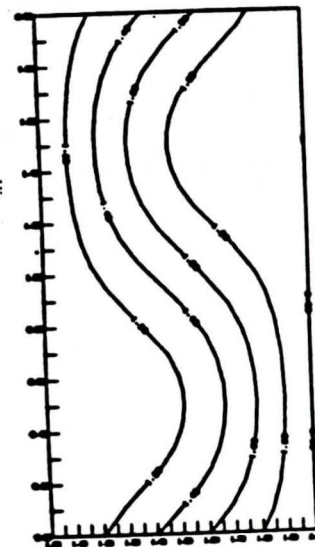
$\gamma = 0.25$  $\gamma = 0.50$  $\gamma = 0.80$  $\gamma = 1.00$ 

Figure 5.2

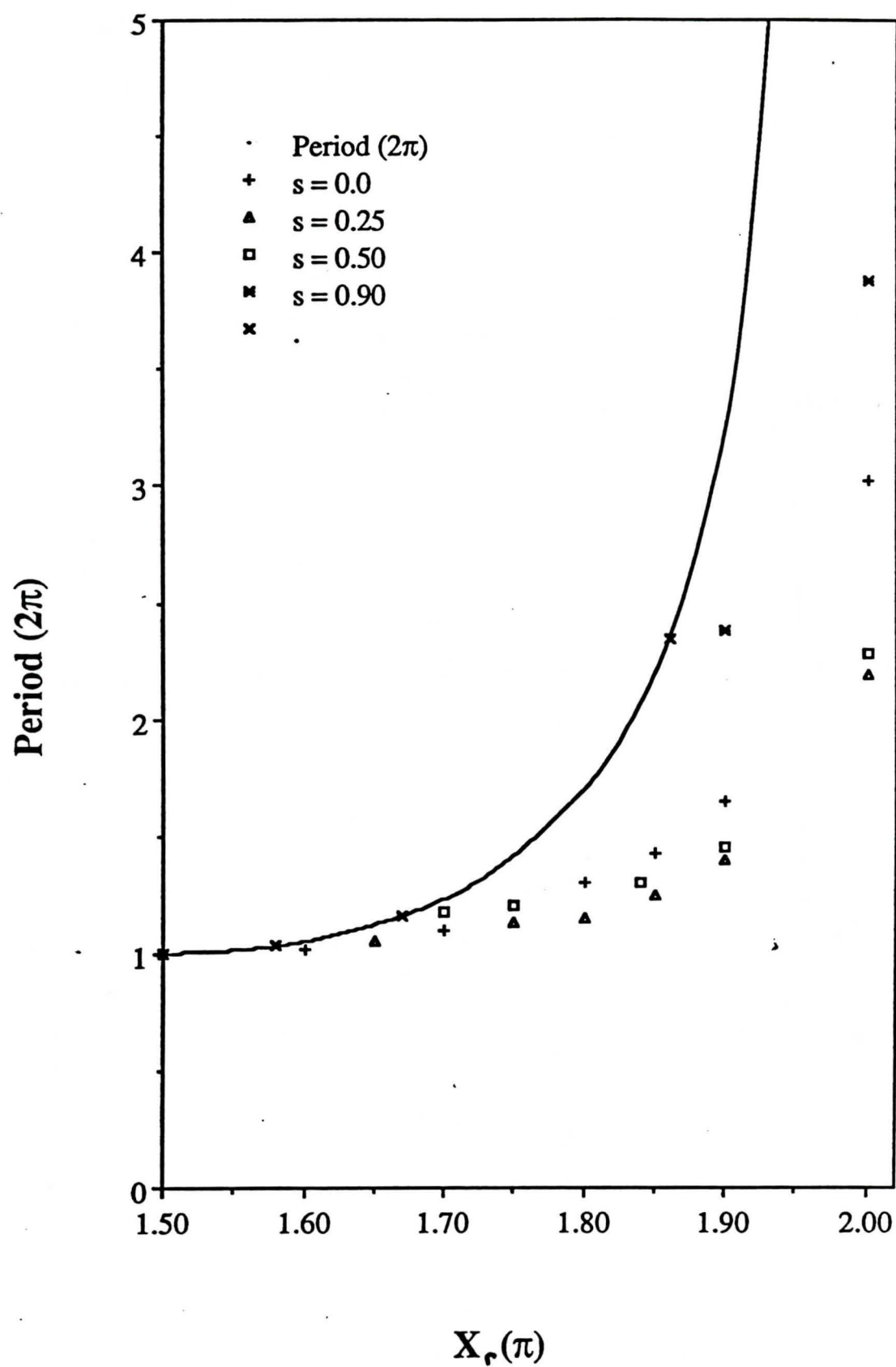


Figure 5.3

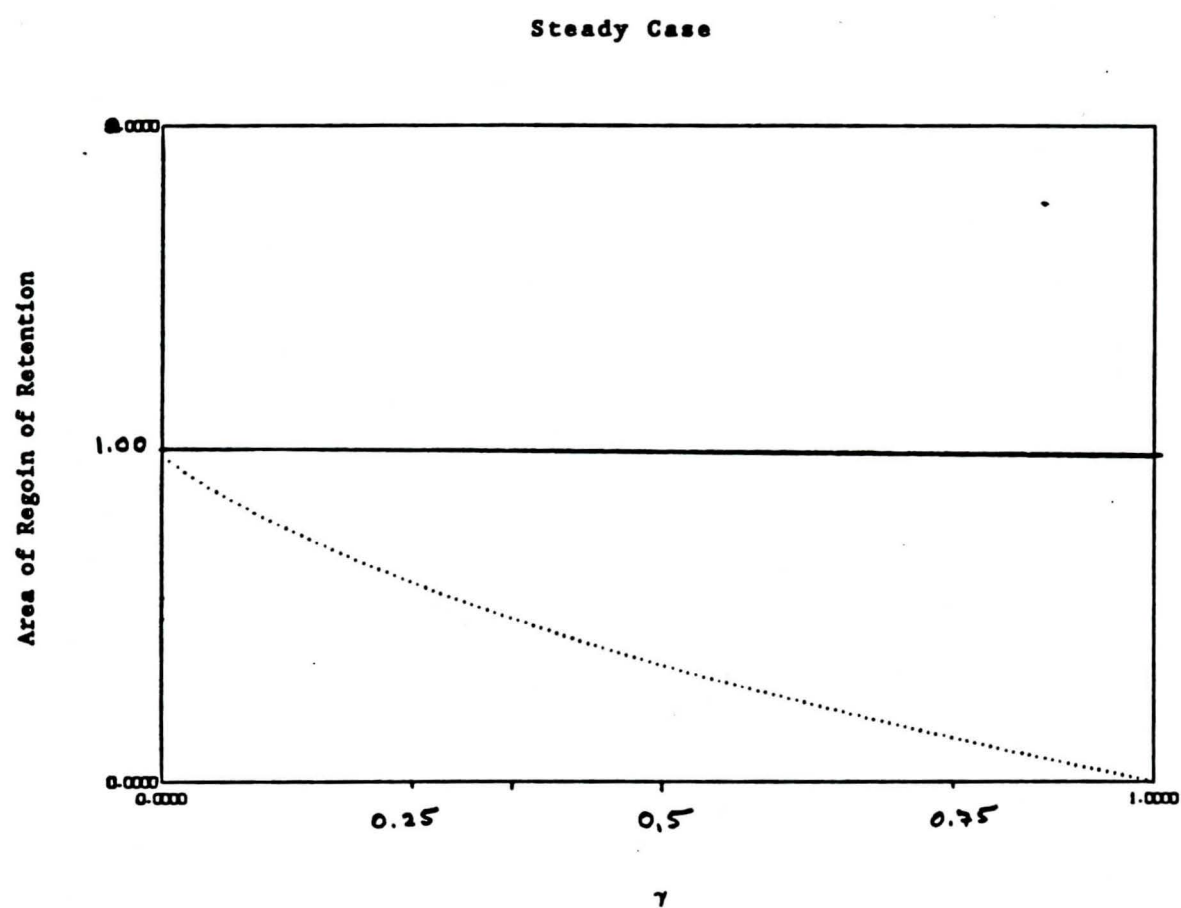


Figure 5.4

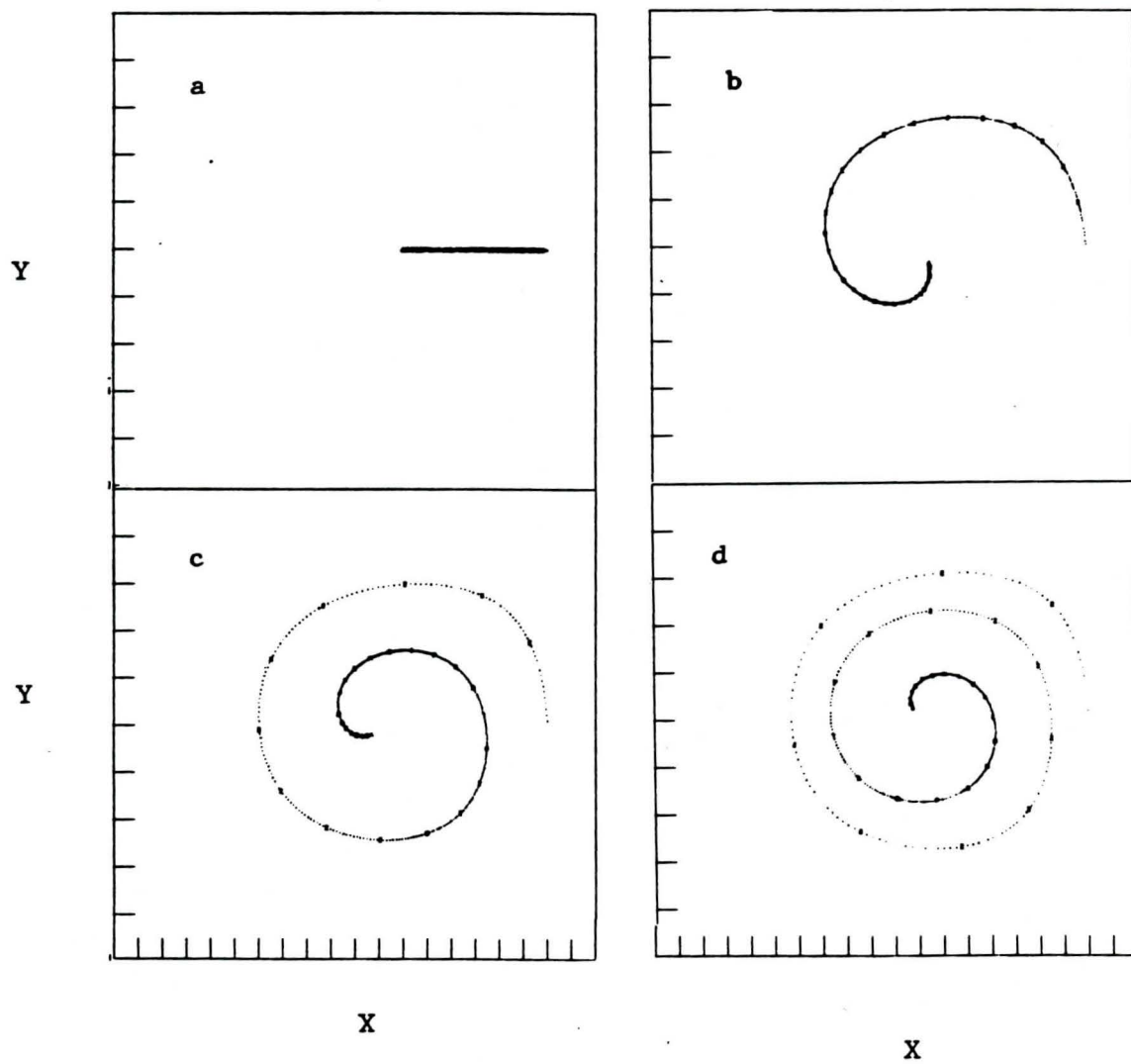


Figure 5.5

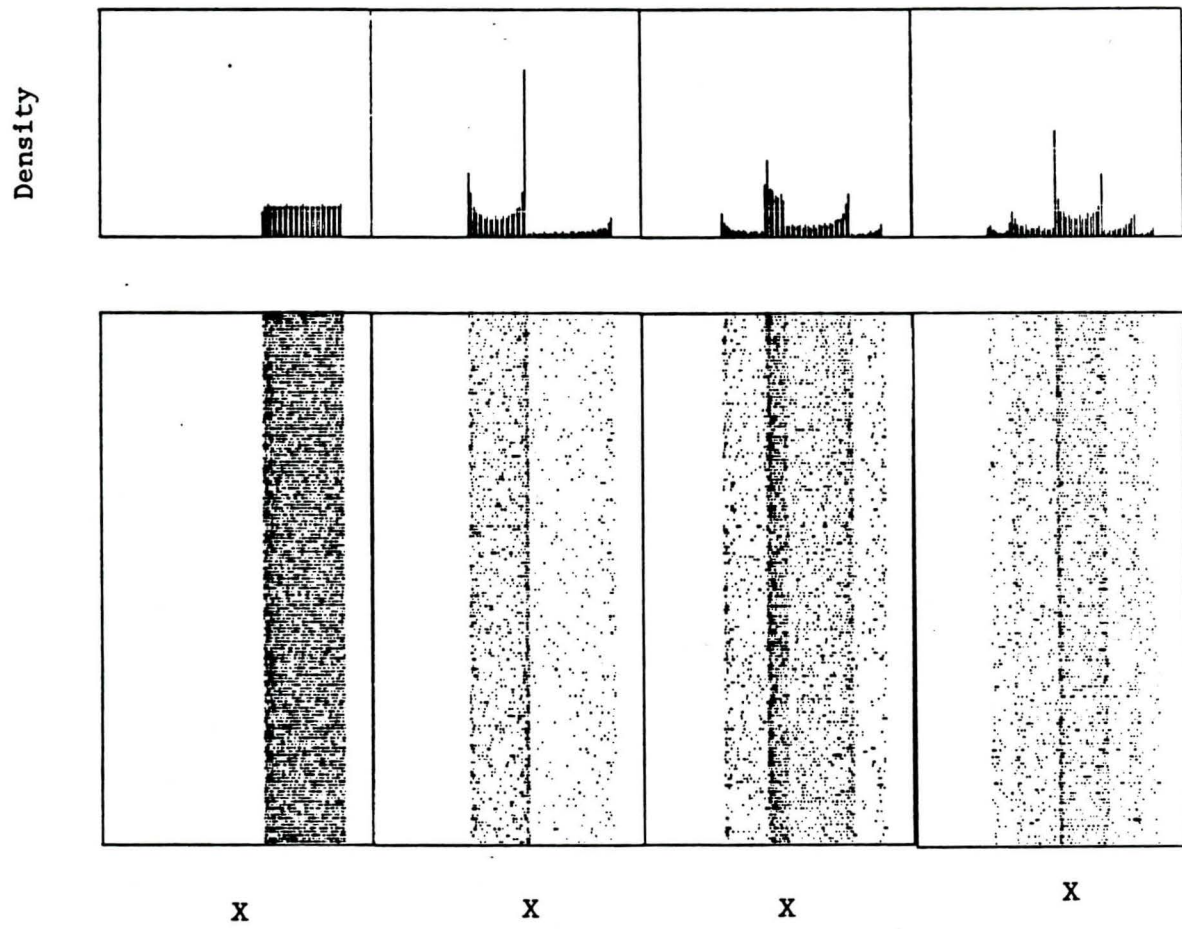


Figure 5.6

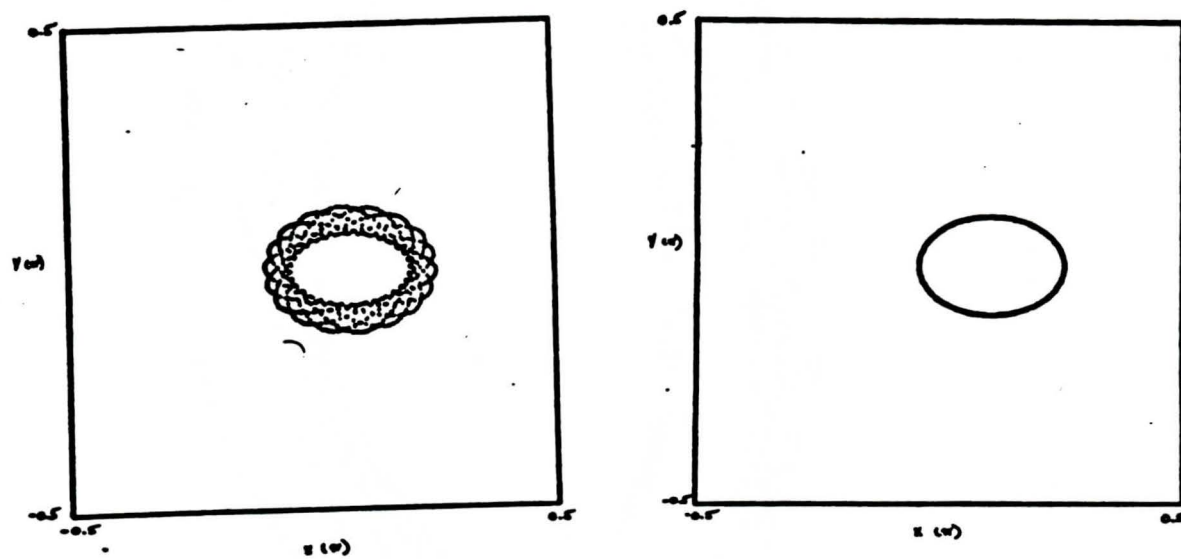


Figure 5.7

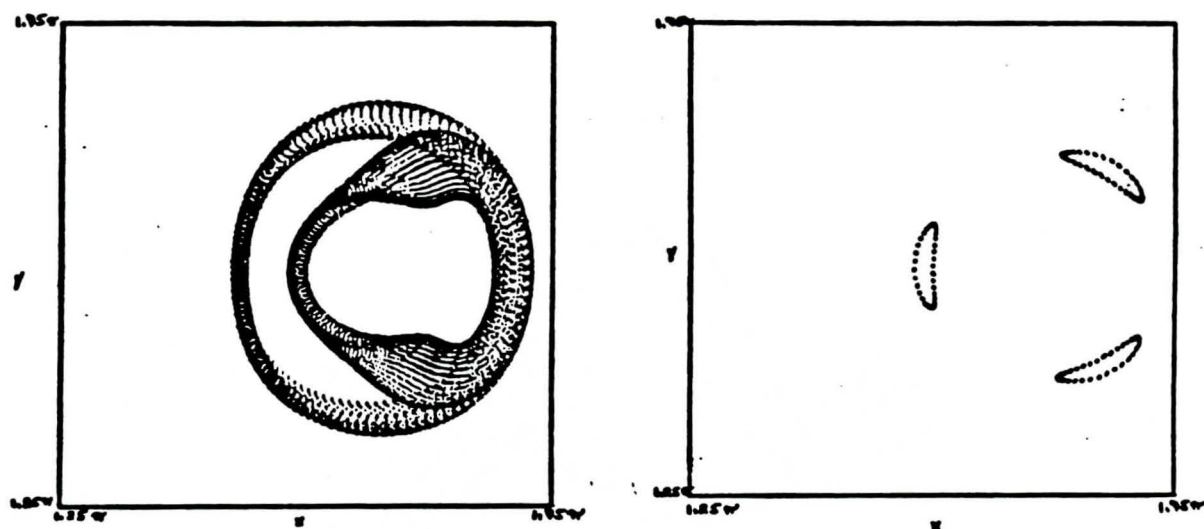


Figure 5.8

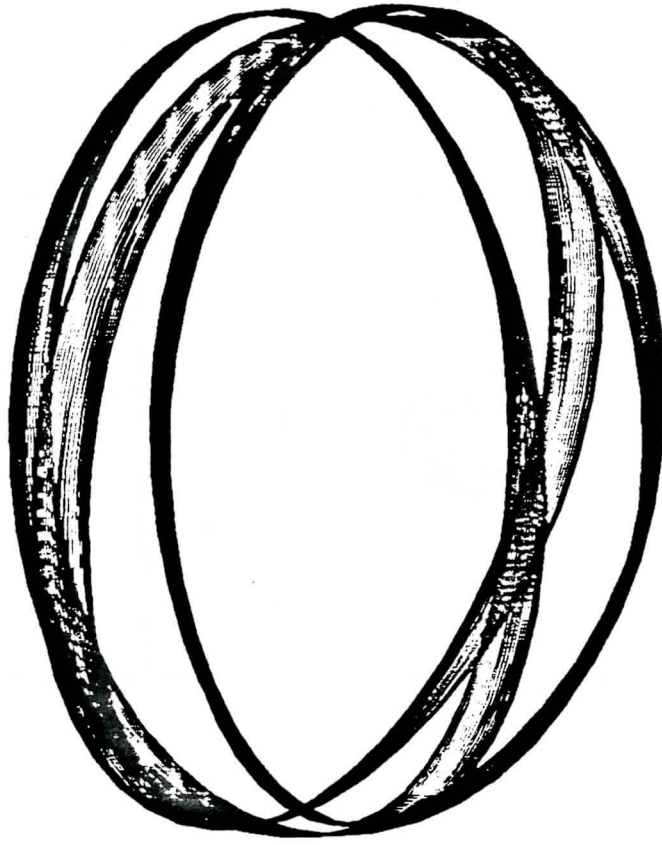
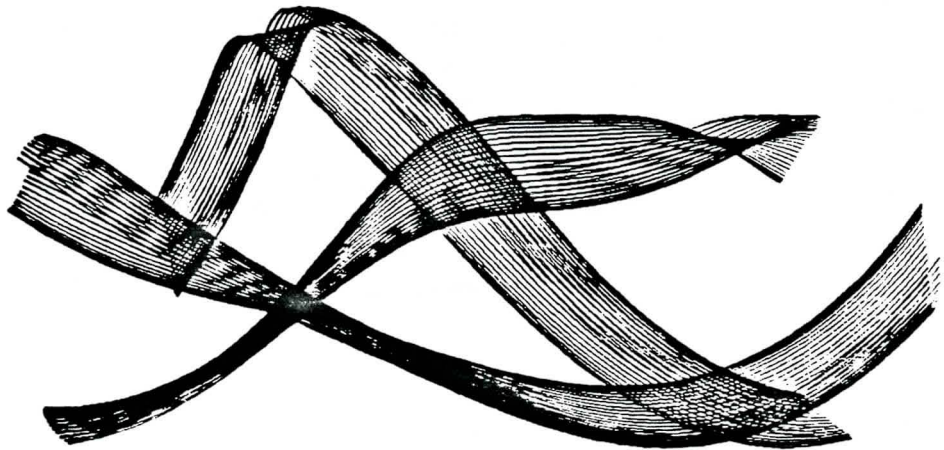


Figure 5.9



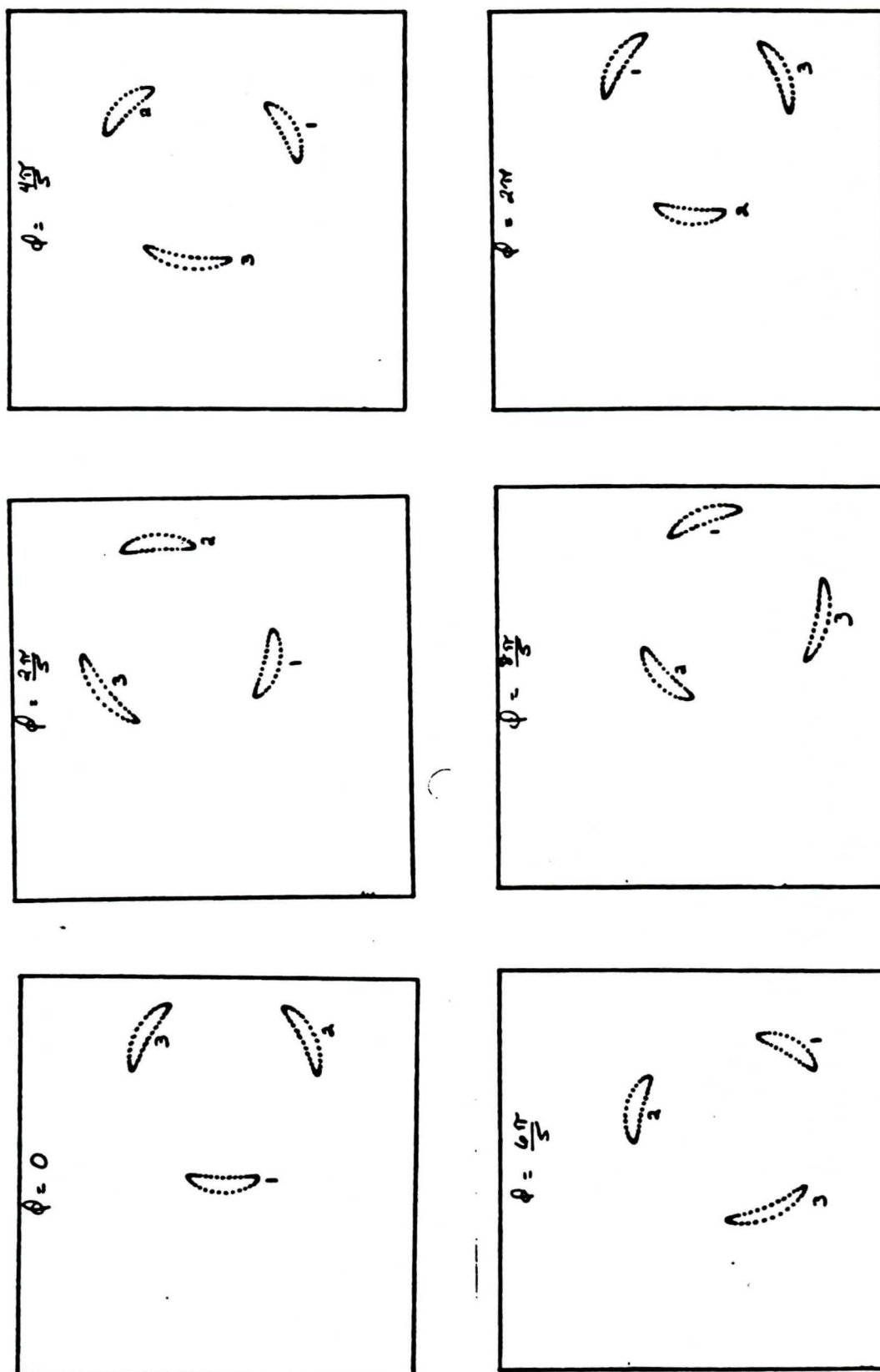


Figure 5.10

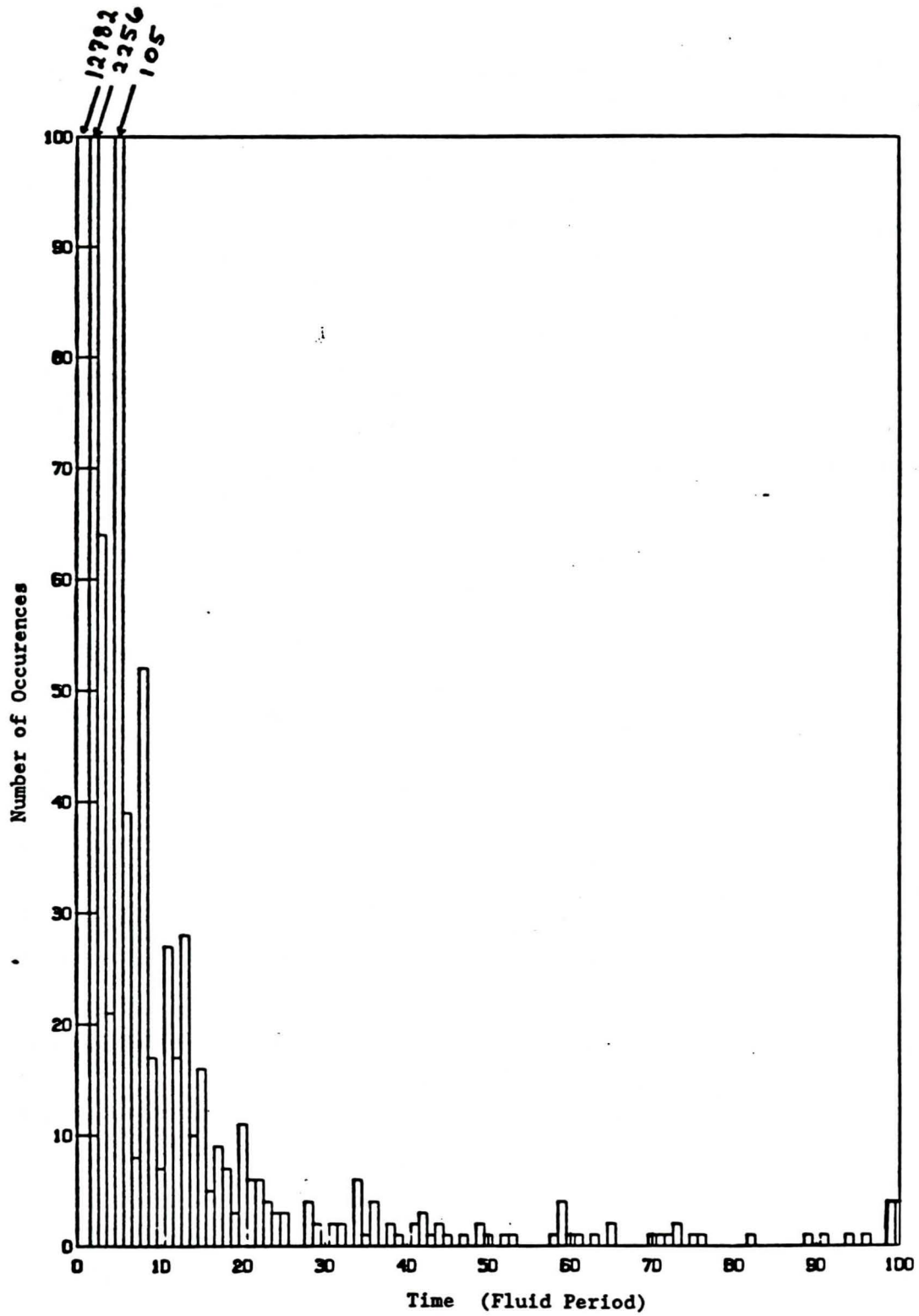


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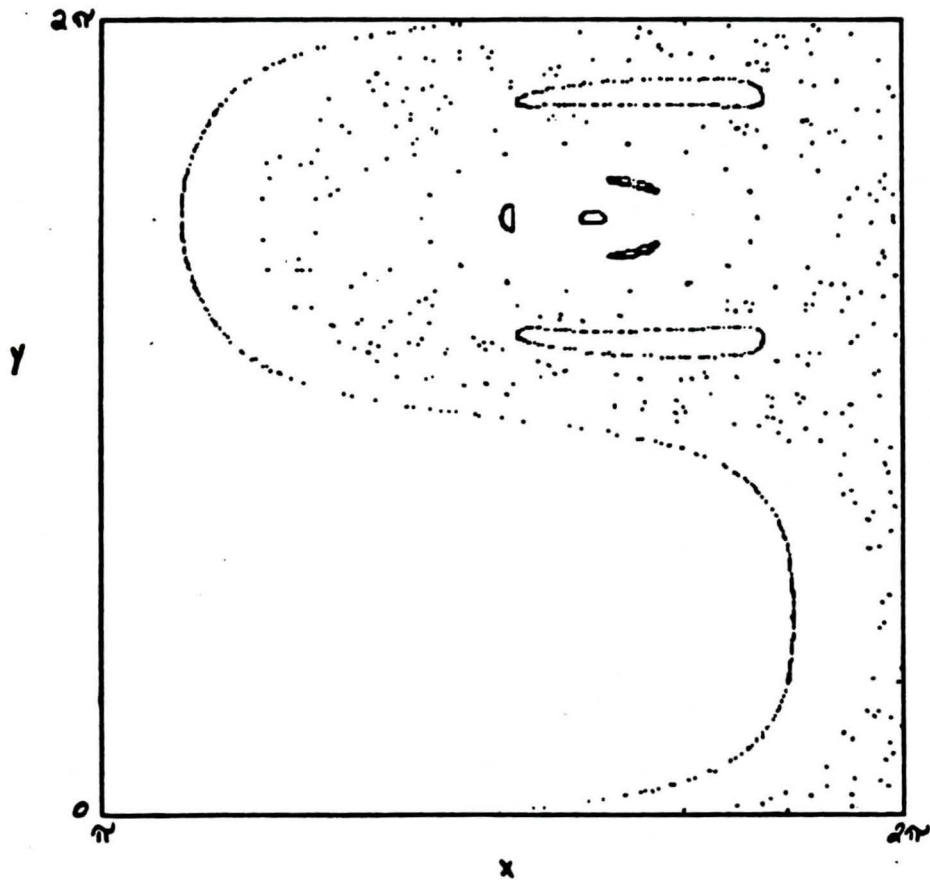
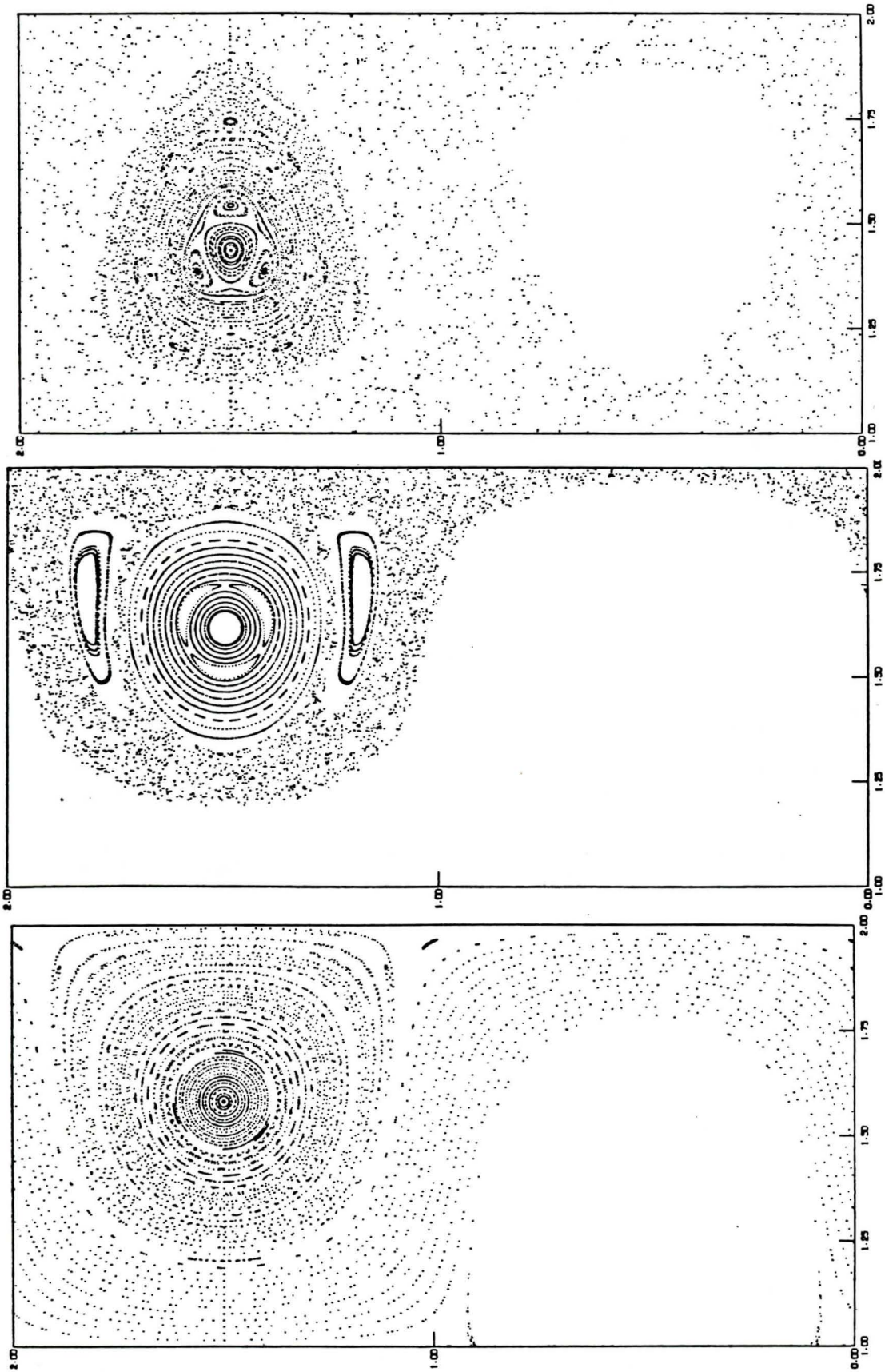


Figure 5.12



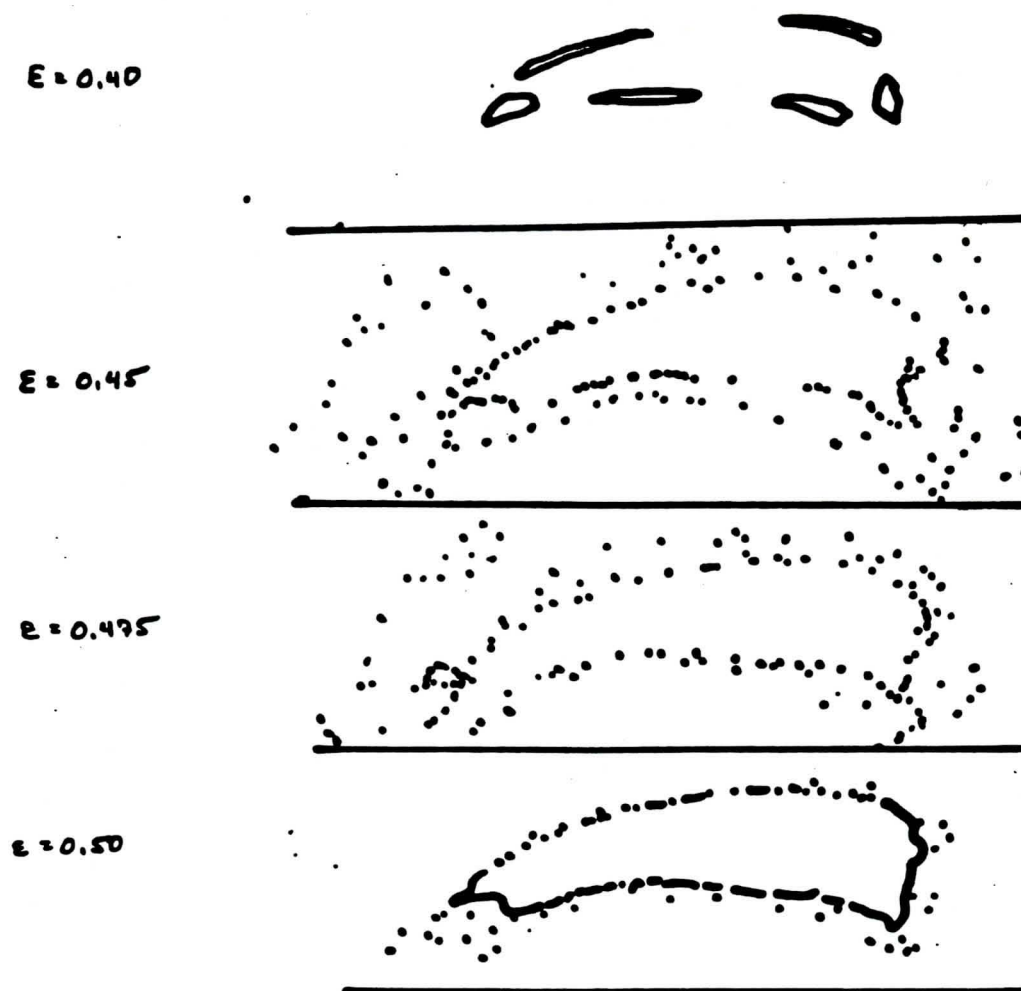


Figure 5.14

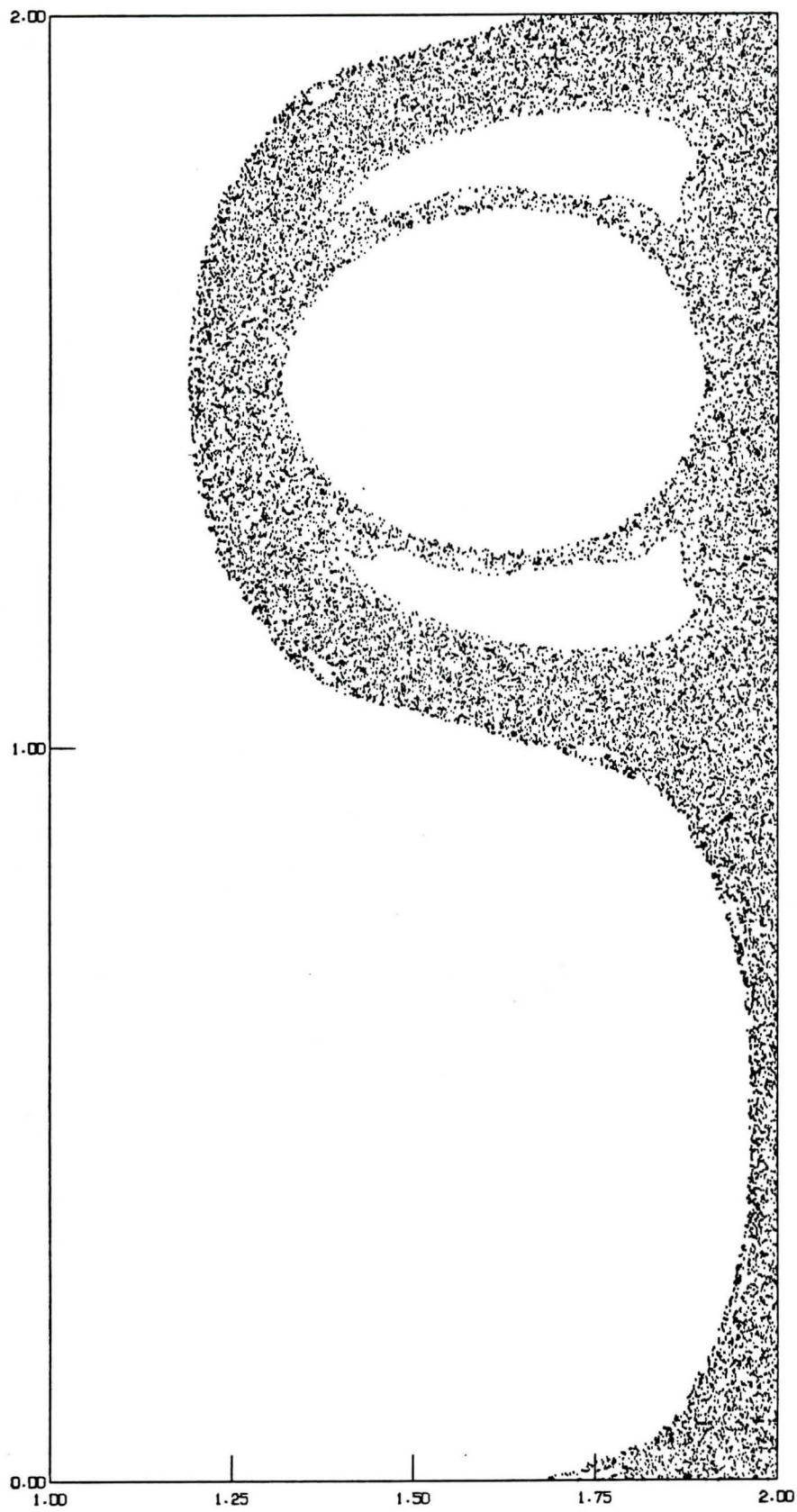


Figure 5.15

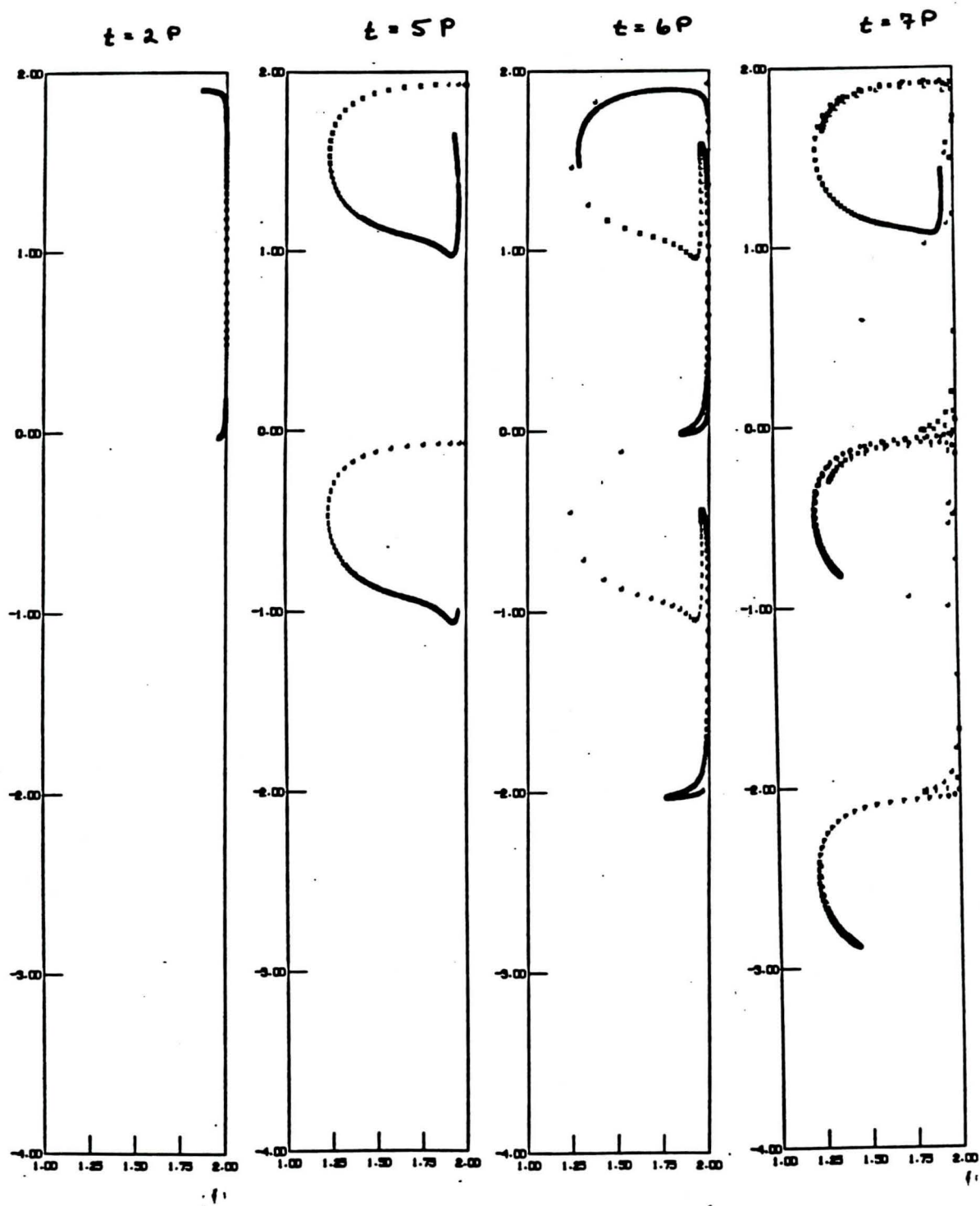


Figure 5.16

$t = 21 P$

228

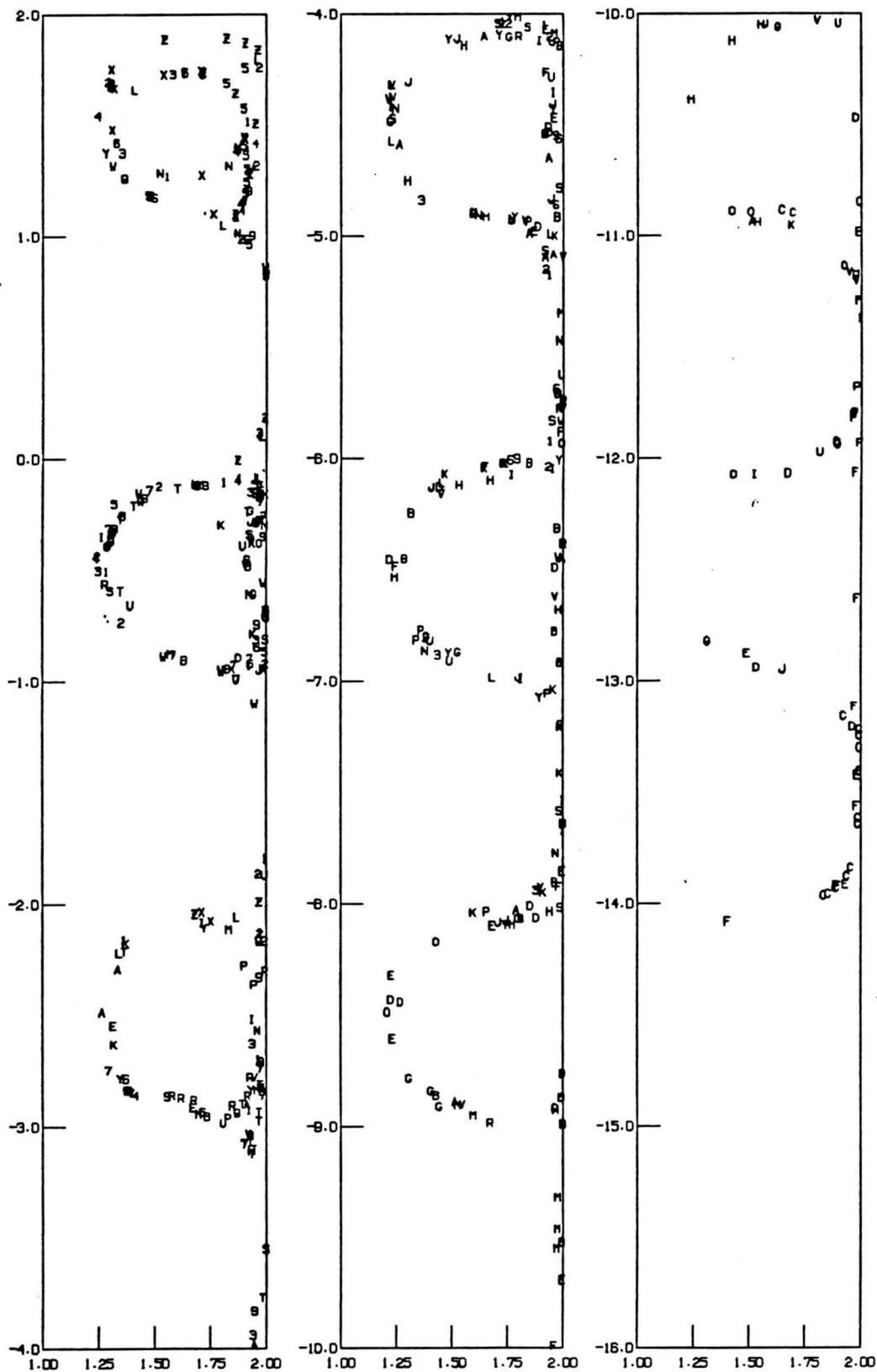


Figure 5.16

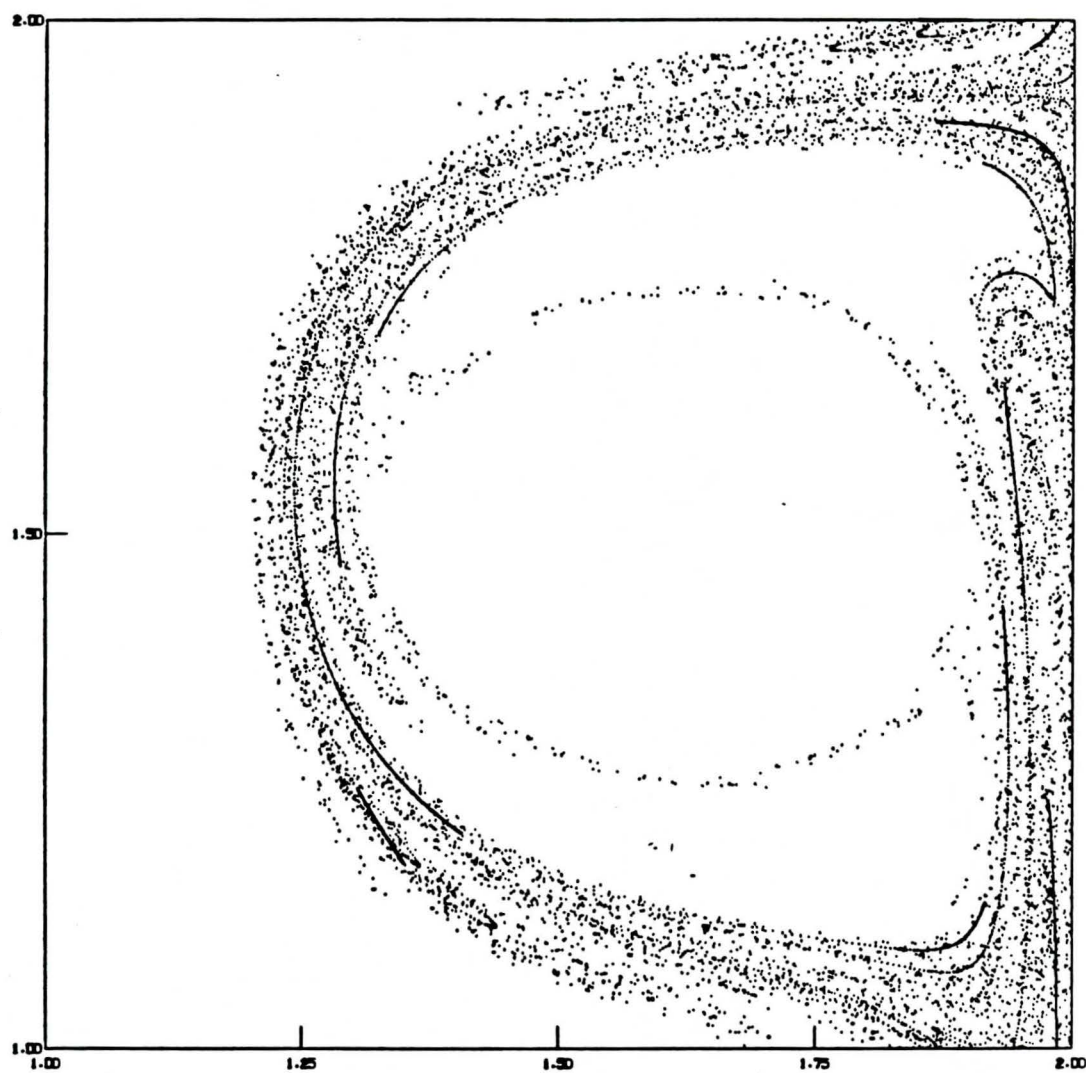


Figure 5.17

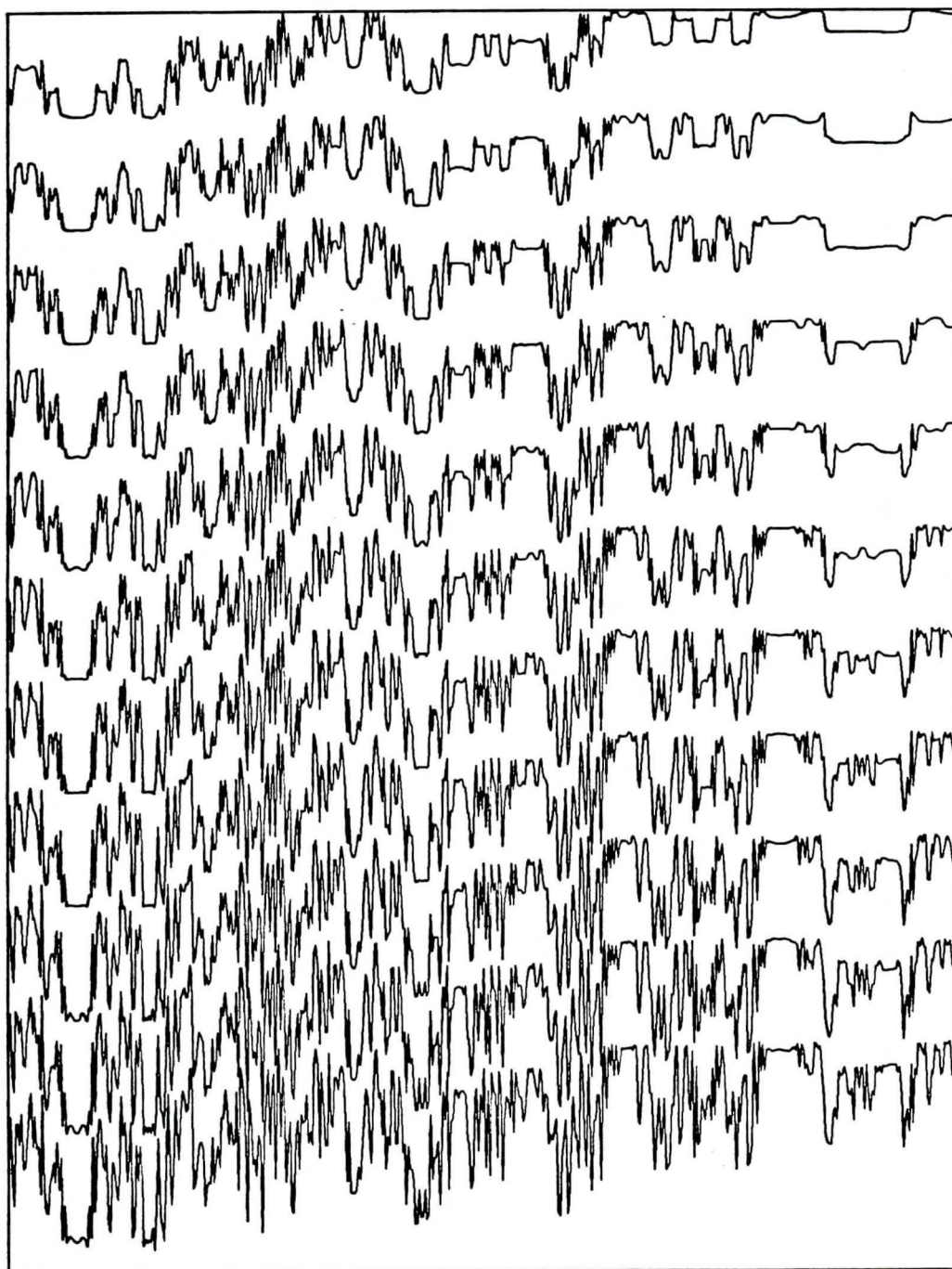


Figure 5.18

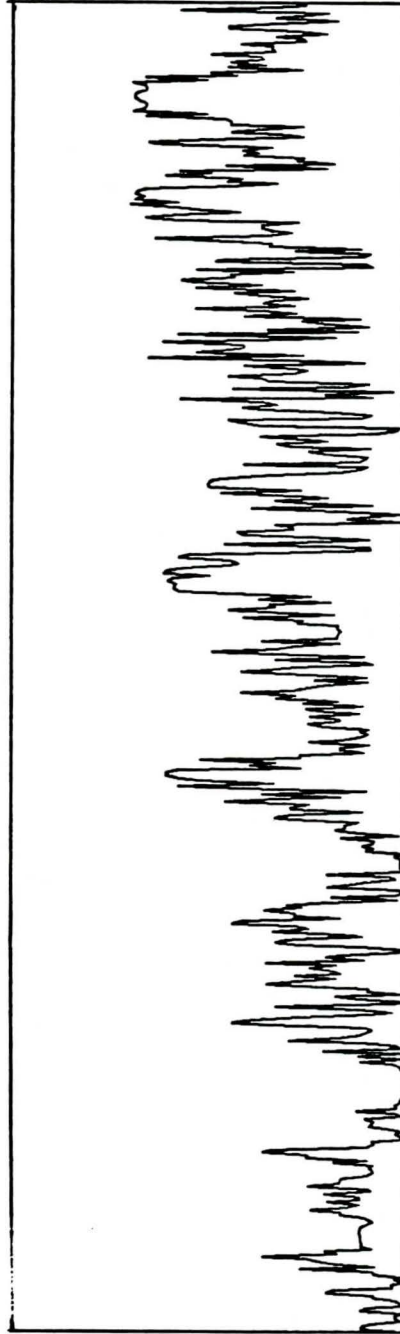


Figure 5.19

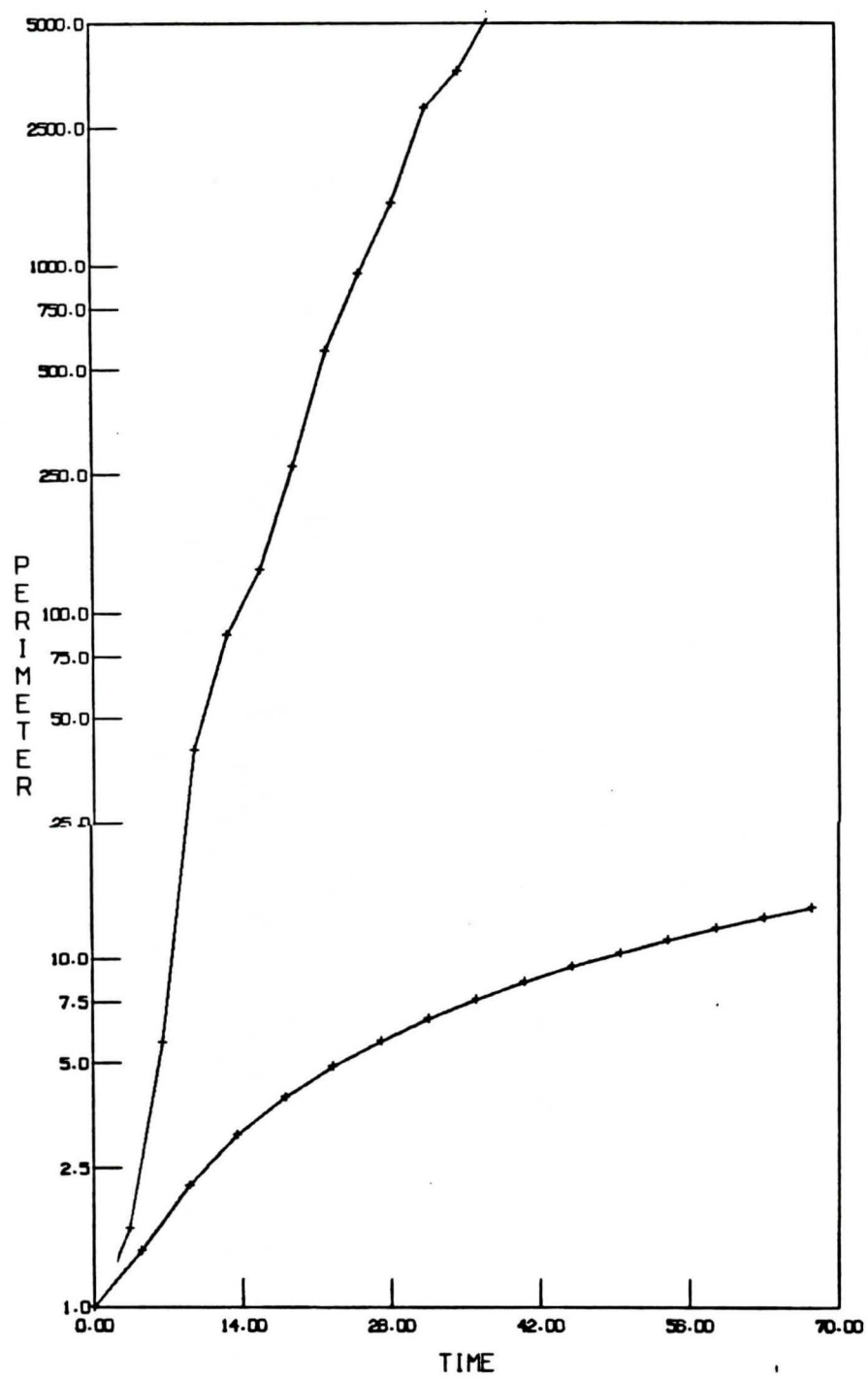


Figure 5.20

Appendix 3: An Class of Iterative Maps

In this appendix, we present a class of two-dimensional area preserving maps which simulate a surface of section for flows of the type discussed in Chapter 5. While it is not possible to predict a single chaotic trajectory, the general topological features of the phase space associated with a set of parameter values may be determined by integrating the equations of motion for a variety of initial conditions and studying the nature of the resulting trajectories. It was in this way that the surfaces of section presented in Chapter 5 were generated. Large amounts of computer resources are required to generate these sections; the motivation behind deducing a map is that such a map provides a method capable of quickly surveying the phase space for a variety of parameter values.

For the time-dependent flow described by Equations 5.5 and 5.15, a map accurate near the fluid stagnation point may be constructed as the composition of 4 simple maps; a twist, a vertical displacement, a lesser twist, and a second vertical displacement equal in magnitude to the first. The product of these maps is area preserving since each of the maps are area preserving individually. The amplitude of the flow is reflected by the strength of the twists while of oscillatory nature of the flow is shown in the difference between the two twists; the average magnitude of the modulation is added in the first twist and subtracted in the second. The vertical displacements reflect the magnitude of the Stokes velocity, v_s and the period of the flow, P . For clarity, we specify the twist maps (T_1 and T_3) in

polar coordinates, and the vertical displacements (T_2 and T_4) in cartesian coordinates.

$$T_1 (r, \theta): \quad \begin{aligned} r' &= r \\ \theta' &= \theta + \tau_+ F(r)/r \end{aligned} ,$$

$$T_2 (x, y): \quad \begin{aligned} x'' &= x' \\ y'' &= y' - \delta_y \end{aligned} ,$$

$$T_3 (r, \theta): \quad \begin{aligned} r''' &= r'' \\ \theta''' &= \theta'' + \tau_- F(r'')/r'' \end{aligned} ,$$

$$T_4 (x, y): \quad \begin{aligned} x'''' &= x''' \\ y'''' &= y''' - \delta_y \end{aligned} ,$$

where

$$\tau_+ = \frac{P}{2} (1 + \bar{\epsilon})$$

$$\tau_- = \frac{P}{2} (1 - \bar{\epsilon})$$

$$\delta_y = \frac{v_s}{A} \frac{P}{2}$$

and

$$\bar{\epsilon} = \frac{2}{\pi} \epsilon .$$

The map is then

$$T(x, y) \quad \equiv \quad T_4 \cdot T_3 \cdot T_2 \cdot T_1 .$$

A more compact form is obtained by considering the map as acting on the complex variable z where

$$z = x + i y.$$

In this case, this map composed of two complex maps

$$T^{\pm}(z) = \exp\{i r_{\pm} F(|z|)/|z|\} z - i \delta_y,$$

as

$$T(z) = T^+ \cdot T^-.$$

A particular map of this class is determined by specifying the velocity function, $F(r)$. We examined several functions, the simulations shown in the figures were obtained with

$$F(r) = \begin{cases} \sin r & r \leq \frac{\pi}{2} \\ 0 & r > \frac{\pi}{2} \end{cases}$$

where we have made the identification $r = |z|$. In each of the cases shown, the initial conditions are along a line of constant y , equally spaced in x . In Figure A3.1, the Stokes velocity is zero and the map reproduces the fluid streamlines of the equivalent flow. The correspondence between the mapping and the full flow is best where the streamlines agree, the poorest correspondence occurring at large r . As δ_y is increased, behaviors in qualitative agreement with the those of the full system are observed.

The major difficulty we found in matching the map to the flow of Chapter 5 was following the particles which passed through the boundaries in the mapping. In the illustrations periodic top and bottom conditions were imposed. Specifically, if at any time

$$y < -\frac{\pi}{2}$$

then

$$y \rightarrow y + \pi$$

$$x \rightarrow -x$$

These top and bottom conditions correspond to the stacked cells described in Chapter 5 where two vertically adjacent cells have been folded onto each other with a reflection. Hence only a single square cell is shown. If a particle is displaced horizontally from the original "cell", that is if

$$|x| = \pi/2,$$

then the calculation is halted. With F9r) as above, the side boundaries are not crossed.

The mapping reflects the flow most accurately when the displacements due to the rotations are small relative to the cell height yet large relative to δ_y . This implies small values of P and v_s . Even in this case, particles which are far from their stagnation point sample regions of the mapping which do not accurately reflect the flow; this is especially true for particles which pass outside the circle $r = \pi/2$.

Qualitatively, the behaviors present in the flow are reflected in the mapping. For v_s equal to zero, the mapping yields circular trajectories as shown in Figure A3.1. For nonzero v_s , there exists a region of retention the area of which decreases as v_s increases. Some of the particles in the chaotic fallout region behave

differently than any particle in the full flow due to the reasons mentioned above. Figure A3.2 shows typical behavior for a moderate value of v_s ; the chaotic fallout region contains integrable island chains, as in the full flow. Motion resembling that on a twisted torus is shown in Figure A3.3. In this case, bounded chaos is present as in the full flow. Finally, in Figure A3.4, a case with relatively large values of v_s and ϵ is shown. The large void region is the region of retention. The smaller lacuna represent regions of integrable fallout. Density variations in the chaotic sea are also apparent in the figure. This is evidence of strange accumulator behavior discussed in Appendix 4.

COMPLEX MAP
AMP = 0.3183
PERIOD = 1.0000
VSTOKE = -0.0100
EPS = 0.1000
NPOINT = 0
X0 = 0.0100 0.1047
Y0 = 0.0000 0.0000

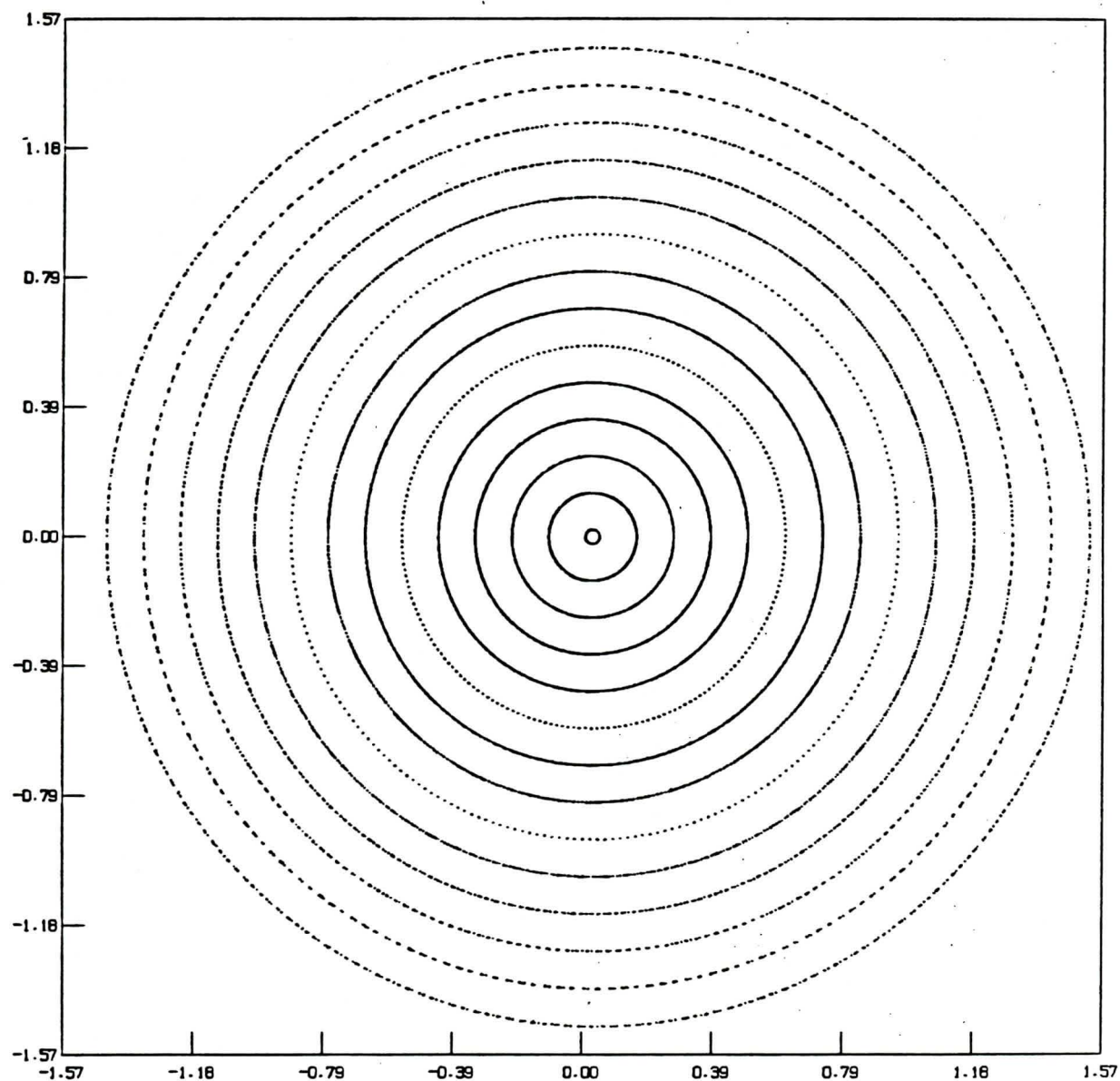


Figure A3.1

COMPLEX MAP
AMP = 0.3183
PERIOD= 1.0000
VSTOKE= -0.3000
EPS = 0.1000
NPPOINT= 0
XD = 0.3047 0.1047
YD = 0.0000 0.0000

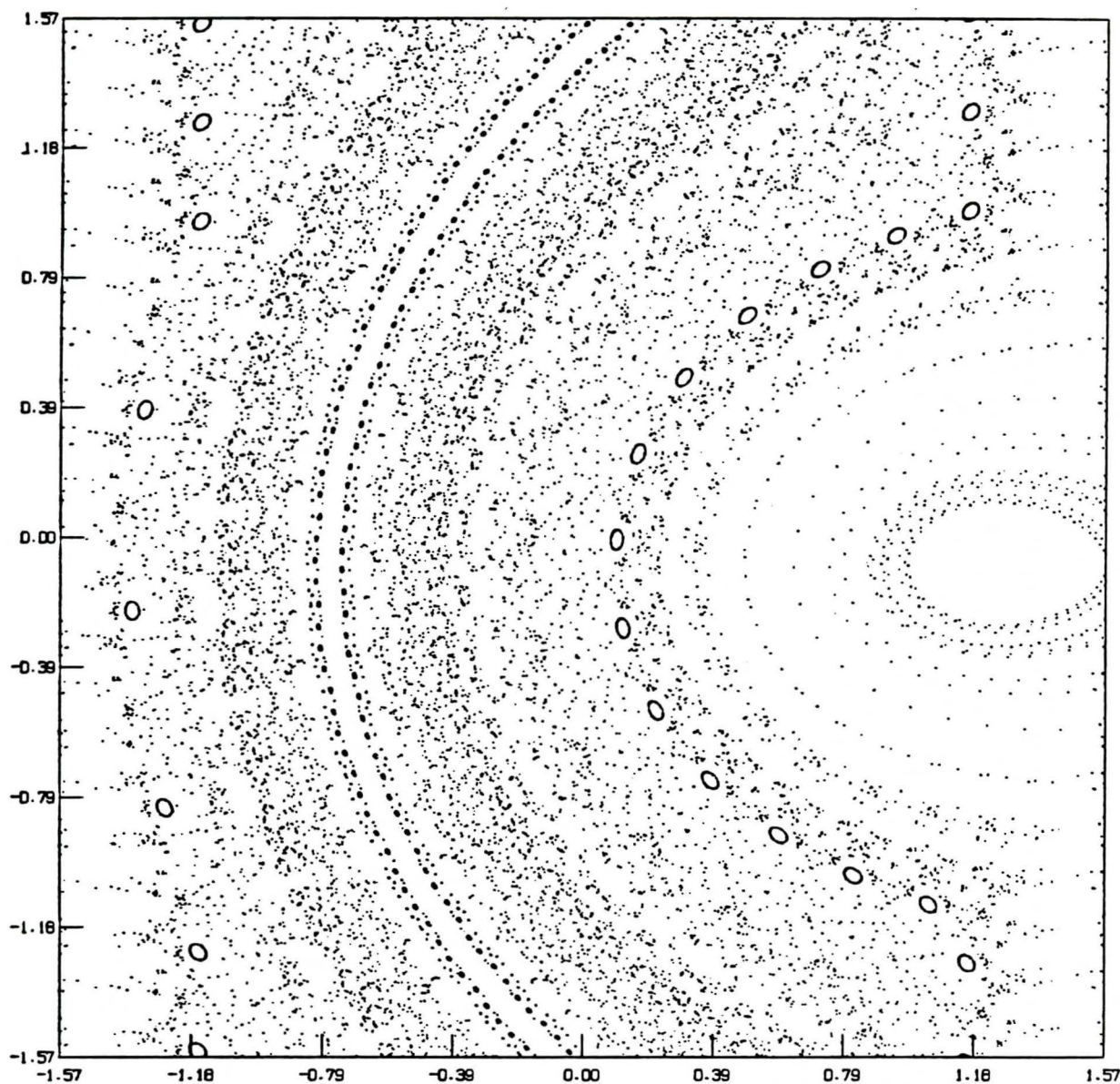


Figure A3.2

COMPLEX ANAMAP
TIME:
COMPLEX MAP
AMP = 1.0000
PERIOD= 4.5000
VSTOKE= -0.3000
EPS = 0.2000
NPPOINT= 0
XD = 0.3047 0.1047
YD = -0.1500 0.0000

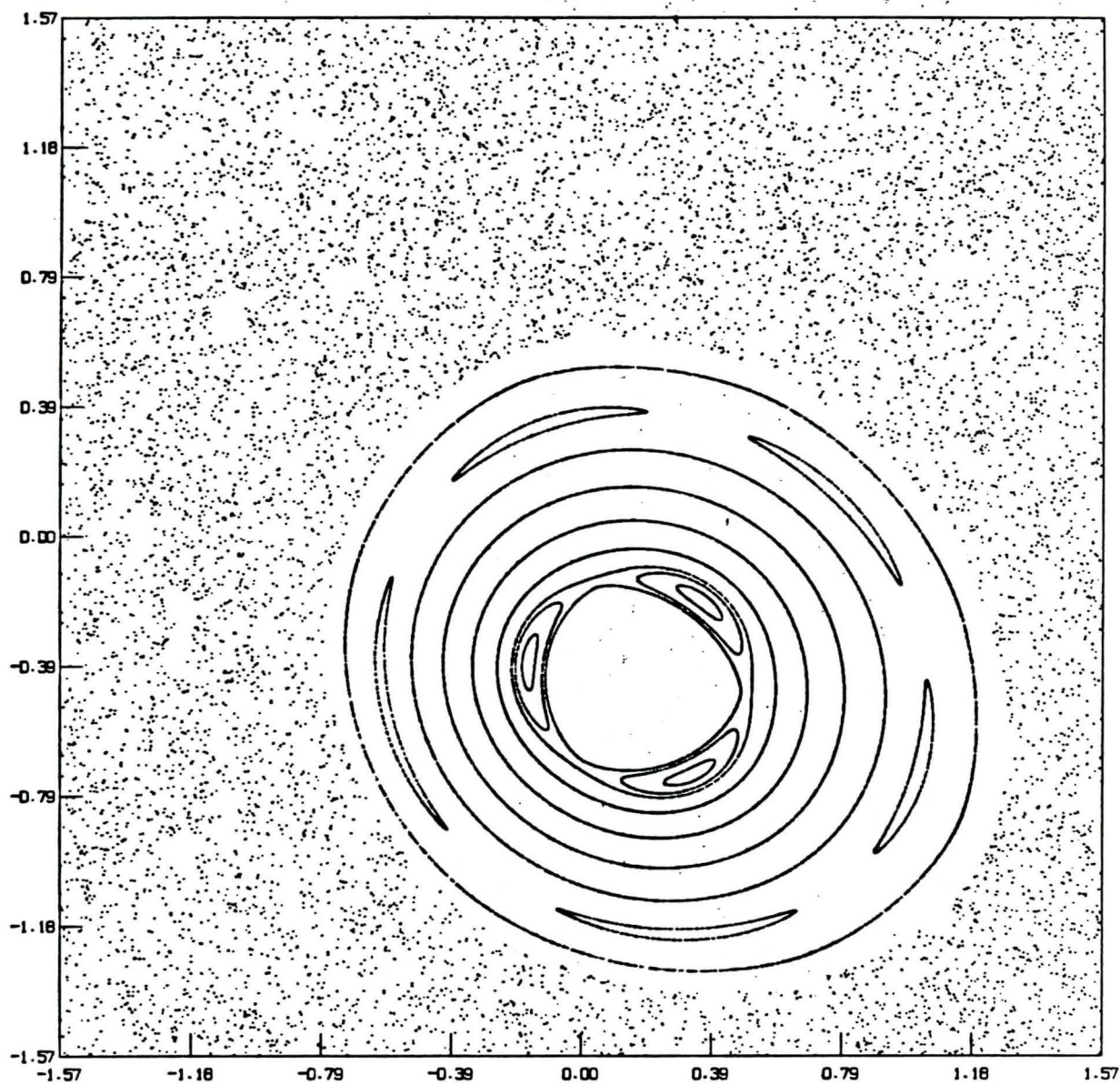


Figure A3.3

COMPLEX MAP
AMP = 1.0000
PERIOD = 4.5000
VSTOKE = -0.5000
EPS = 0.2000
NPPOINT = 0
X0 = 0.5236 0.1047
Y0 = 0.0000 0.0000

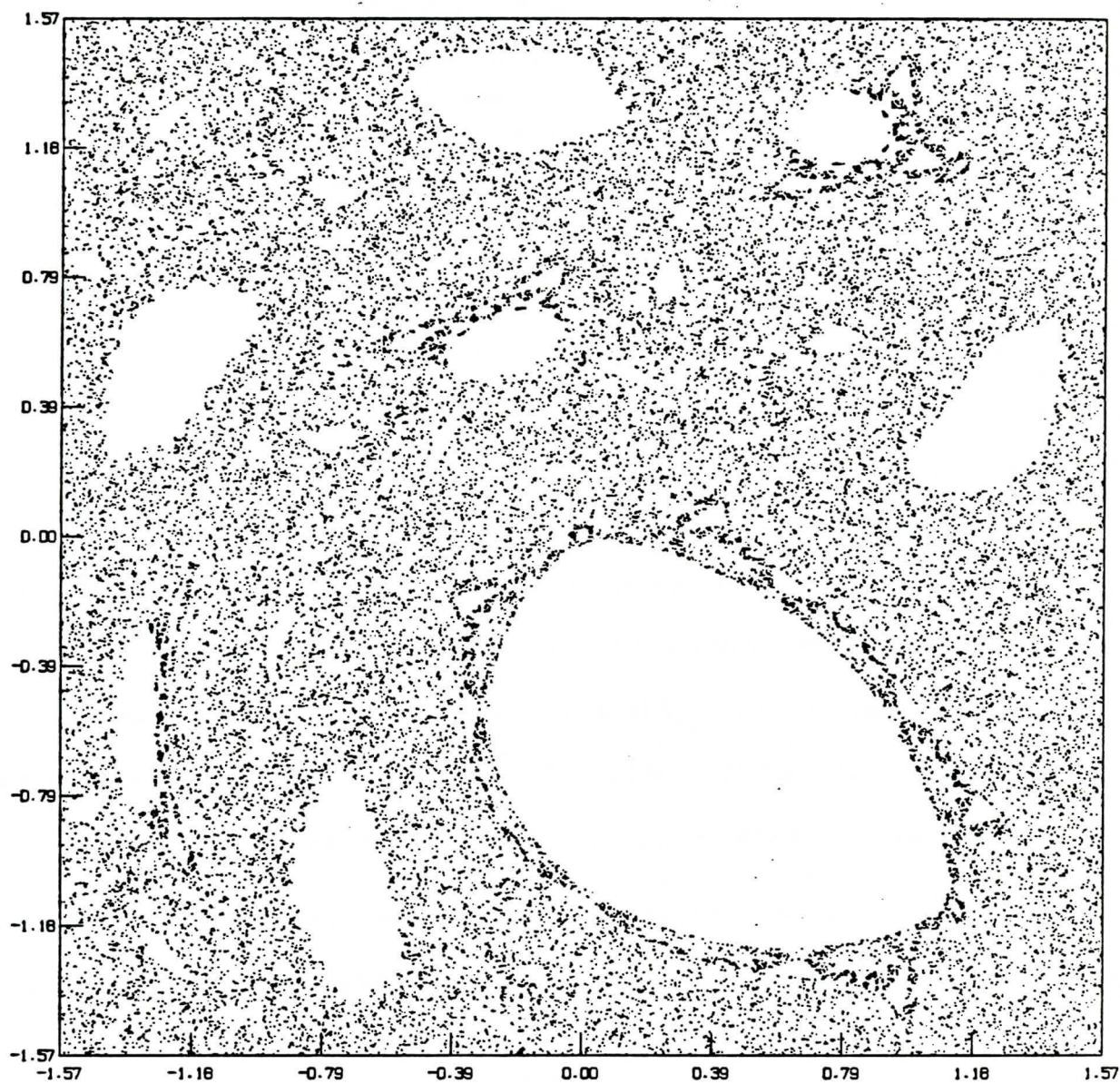


Figure A3.4

Strange Accumulators^a**L.A. SMITH^b AND E.A. SPIEGEL^c****^b Department of Physics****Columbia University****New York, New York 10027****and****NASA Goddard Space Flight Center****Institute for Space Studies****New York, New York 10025****^c Department of Astronomy****Columbia University****New York, New York 10027**

In astronomy, we see many irregular structures manifested in the form of strong inhomogeneities of physical variables like density, temperature, or magnetic field. The appearance of strong concentrations of these variables, in the presence of what frequently must be highly turbulent conditions, poses an interesting question whose answer probably lies in the nature of turbulence itself. Indeed, G.I. Taylor, seventy years ago, wrote that turbulence is the strong concentration of vorticity. How does this work?

Fluid dynamicists often refer to a mechanism that they call Batchelor-Prandtl expulsion to explain inhomogeneities; the nature of this process has recently been clarified by Rhines and Young¹. Solar physicists know of the process as it applies to the development of magnetic inhomogeneities on the solar surface². However, no unanimity seems to exist among solar physicists on the explanation of the fine-scale structures on the sun³. There even seems to be disagreement about the proper description of the topology of field lines.

Dynamical systems theory shows us a simple way to look at field lines that may teach us about their structures. Though the approach does not explain how such fields arise in hydromagnetics (MHD), it may nevertheless be valuable in thinking about the possibilities that confront us when we look at a complicated situation like that of the solar surface.

Consider a vector field $B(x, t)$ whose structure we want to look at. This may be a magnetic or vorticity field whose field structure may be too complicated to pull out of numerical solutions of the MHD equations (for example). Thus, this is an example of how the theoretical developments of chaos theory may help in astrophysics by adding to the ways we have of thinking about problems, even when the data may not be good enough to confirm or deny the presence of chaos.

Let X be a function of an independent variable s such that

$$\frac{dX}{ds} = B(X(s), t) \quad (1)$$

for fixed t . If B should happen to be a velocity field, we would identify s and t and integrate equation 1 to get the particle paths, as has been done for fluid flows in two dimensions^{4,5}. However, if B is some other field, we can look at a snapshot of it by regarding t as a parameter that we hold fixed during a run in which we study equation 1 as a dynamical system with s as the "time." The trajectory in X -space will give us a picture of the field. We can even go further and cut the X -space with a plane and simply plot the points where the trajectory pierces it. By studying typical maps of the plane into itself, we can get an idea of the possibilities that lie in store with real fields, B .

Perhaps the most typical, typical map is the standard map ⁶:

$$x_{i+1} = x_i + k \sin 2\pi y_{i+1} ; y_{i+1} = x_i + y_i \quad (2)$$

This is an area-preserving map that seems a qualitatively appropriate choice when (the solenoidal) B does not vary much in the direction normal to the surface section. For given k (not too large) and for suitable initial coordinates, this map gives rise to regular behavior in the form of periodic orbits lying on tori. The cross sections of these tori in the surface of section form island chains that are like the cat's-eyes of the fluid dynamicist. Between the islands are saddle points, in the neighborhood of which there are smaller islands. And so on.

A trajectory that gets in amongst the small islands will go for many twists before reemerging into the chaotic sea. This sojourn is responsible for long-range correlations of the successive points on the orbits of equation 2^{7,8}. Meiss and Ott⁹ have modeled the tendency of an orbit to linger in the reefs, while McKay, Meiss, and Percival¹⁰, among others, have studied the mechanism of escape from them.

Our own naive point of view about the long times spent by particles in the reefs of the map is to liken these regions to the attractors of dissipative theory. One of the simplest examples of a dynamical system with an attractor is the Landau equation for $A(t)$:

$$\frac{dA}{dt} = A - A^3 . \quad (3)$$

This system has attractors at $A = \pm 1$ and a repeller at $A = 0$. If we differentiate this system once and substitute, we get

$$\frac{d^2 A}{dt^2} = -\frac{\partial V}{\partial A} , \quad (4)$$

where

$$V = -\frac{1}{2}A^2(A^2 - 1)^2. \quad (5)$$

The original system (equation 3) is contained in equations 4 and 5 if we impose suitable initial conditions.

Where the original Landau equation had either attractors or repellers, the new system (which is Hamiltonian) has saddle points. In the neighborhood of these points, the motion is very slow. We call sets of these points accumulators because a representative point moving in this system will tend to spend more time in the neighborhood of these points. Hence, the points in an orbit will tend to accumulate there. In systems with stochasticity and island chains, we can have quite a complex of such points; together, these make an accumulator with very fine texture. The presence of such complicated accumulators, we suggest, gives rise to strong inhomogeneities in field structures.

In the left panels of the accompanying figure (FIGURE 1), we show the result of a long numerical integration based on the map of equation 2. The calculation was done on a finite ($2^{20} \times 2^{20}$) mesh that preserves the 1-1 character of the map of equation 2, as in the work of Rannou¹¹ (see Miller and Prendergast¹² for use of "noise-free" methods in dynamical astronomy). The result of this method (with a large, but finite number of grid points) is that every orbit calculated is periodic. Indeed, any deterministic simulation on a digital computer tends, in finite time, to a periodic orbit. If we are using the result to visualize field lines, this means that we are computing only closed field lines. In this case with an explicit grid, phase volumes are exactly conserved.

Figure 1 shows coarse-grained histograms of the number of visits to each grid square obtained in two different simulations. The panels on the left show histograms from a periodic 2,764,949-iteration orbit for the numerically one-to-one standard map with $k = 1.25$, the lower panel being a blow-up on a finer grid. The panels on the right similarly illustrate a double-precision calculation for the Hénon area-preserving, quadratic map (the field is chosen after figure 14 of Hénon¹³). In the Hénon case we have followed the mapping for 2^{27} iterations. The calculations for the Hénon case, unlike those for the standard map, are not one-to-one numerically, hence the structure shown in the right panels may be transient. The range in density shown is a factor of several hundred. Already, from these first results, we can see why the discussants at the solar physics meeting were in some doubt about what to call a flux tube in the solar surface. The notion of flux tube will often be useful only locally, but then the tube may splay out into the surrounding stochastic regions.

The deficiency in this type of study is that it is a little too generic. We want to be able to derive actual maps for real situations. This can be done when the vector field being studied is a real velocity field. Indeed, Chaikin *et al.*⁵ have set up real flows corresponding closely to their maps and have found excellent agreement for the motion of advected particles. The flow in that case involved time-dependent open streamlines. We may also get open streamlines by studying the motion of particles drifting through a convecting fluid^{14,15}. These studies suggest formation of strong inhomogeneities in the particle densities, but the results are, as yet, still limited.

In summation, we can say that this note is a conference publication *par excellence*. We

are suggesting a way for modeling processes that seem too complex for detailed simulation without yet offering any quantitative predictions for astrophysics. Nevertheless, our qualitative results from the numerical experiments on long-period orbits seem to have surprised many of those who have seen them. Thus, it seems worth pointing out our conclusion that complicated accumulators are at the origin of strongly inhomogeneous concentrations of advected fields in flows. These fields need not be passive. Though the fields are dynamic and feed back on the flow, we may still expect the structures to be given by maps, even if we do not know them explicitly. This is ultimately a useful way to think about inhomogeneous fields.

ACKNOWLEDGEMENTS

We are very grateful to B. Bayly, J. Meiss, R. Mackay, and K. Prendergast for illuminating discussions. We look forward to further illumination as we continue to pursue particles around the plane.

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CAPTION

The panels on the left show the visitation histogram for the numerically one-to-one standard map ($k = 1.25$). The lower panel is a blow-up. On the right, simulations with the Hénon¹³ conservative map ($\cos \alpha = 0.240$) are shown. The number of visitations, as indicated by each color, varies slightly from panel to panel for photographic reasons. In the upper left panel, black means no visitation, brown is 1 to 4, blue is 5 to 9, red is 10 to 19, green is 20 to 29, yellow is 30 to 50. White partitions were visited between 51 and 316 times. All histograms are calculated on a 2^9 by 2^9 grid.

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